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CONTENTS

PAGE
PROCEEDINGS 1
THE METHOD OF PONICARÉ 5
ON A CERTAIN CUBIC SURFACE CALLED THE MÖBIUS SURFACE. *By* Professor C. E. Cullis 9
NEW METHODS IN THE GEOMETRY OF A PLANE ARC, I. *By* Professor Syama Das Mukhopadhyaya .. 31
ON A NON-ANALYTICAL POTENTIAL FUNCTION. *By* Professor Ganesh Prasad 39

SUMMARY OF PRINCIPAL MATHEMATICAL JOURNALS:

Mathematische Annalen, Vol. 66, Parts 1—3 43
Journal de Mathematiques (Lionville's Journal), Series 6, Vol. 4, Nos. 1 and 2 49
Annals of Mathematics, Series 2, Vol. 9, Nos. 2 and 3 .. 51

SOCIETIES AND ACADEMIES:

Academy of Sciences, Paris, January—June, 1908 .. 53
Royal Society, London, January—March, 1909 54
London Mathematical Society, January—March, 1909 .. 55
American Mathematical Society, April 1907—April 1908 55

REVIEWS:

A course of Pure Mathematics. G. H. Hardy 57
Traite de Trigonometrie. J. A. Serret 61
The Corpuscular Theory of Matter. Sir J. J. Thomson 62
NOTES AND NEWS 65
NEW PUBLICATIONS 69
LIST OF FOUNDATION MEMBERS 71
LIST OF BOOKS 77
ON A CERTAIN CUBIC SURFACE CALLED THE MÖBIUS SURFACE (PART II). *By* Professor C. E. Cullis .. 83
THE EQUATION OF DIGITS: BEING AN ELEMENTARY APPLICATION OF A PRINCIPLE OF NUMERICAL GROUPING TO THE SOLUTION OF NUMERICAL EQUATIONS. *By* Professor Brajendra Nath Seal .. 99
ON RATES OF VARIATION OF THE OSCULATING CONIC. *By* Professor Syama Das Mukhopadhyaya . 125
ON THE FOUNDATIONS OF THE THEORY OF SURFACES. *By* Professor Ganesh Prasad 131
SPACE AND TIME 135

SUMMARY OF PRINCIPAL MATHEMATICAL JOURNALS:

Mathematische Annalen, Vol. 67, Parts 1 and 2 142

SOCIETIES AND ACADEMIES:

Academy of Sciences, Paris, July—December, 1908 .. 149
German Association of Mathematicians, 1908 150
Royal Society, London, April—June, 1909 150
London Mathematical Society, April—June, 1909 .. 150

REVIEWS:

Introduction to Higher Algebra, Maxime Bôcher .. 152
An Introduction to the Study of Integral Equations, Maxime Bôcher 153
The Mathematical Theory of Electricity and Magnetism, J. H. Jeans 154

PAGE
NOTES AND NEWS 157
NEW PUBLICATIONS 160
ON THE EQUATIONS OF MÖBIUS SURFACES OF ALL PITCHES (PART I). *By* PROFESSOR C. E. CULLIS .. 163
PARAMETRIC COEFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES, I. *By* PROFESSOR SYAMA DAS MUKHOPADHYAYA 187
SUMMARY OF PRINCIPAL MATHEMATICAL JOURNALS:
Mathematische Annalen, Vol. 68, Parts 2 and 3 201
Journal de Mathematiques (Lionville's Journal), Vol. 4, Nos. 3 and 4 204
REVIEWS:
An Introduction to the theory of Infinite Series. T. J. I'A. BROMWITCH 207
Grundzüge der Differential-und Integralrechnung. G. KOWALESWSKI 210
The Ether of Space. SIR OLIVER LODGE 212
SOCIETIES AND ACADEMIES:
Royal Society, London, July—September, 1909 216
London Mathematical Society, July—September, 1909 .. 216
Academy of Sciences, Paris, January—September, 1909 .. 217
American Mathematical Society (Transactions), April—July, 1909 218
SIMON NEWCOMB 219
NOTES AND NEWS 222
NEW PUBLICATIONS 225
LIST OF ADDITIONS TO THE LIBRARY 228
PARAMETRIC COEFFICIENTS IN THE DIFFERENTIAL GEOMETRY OF CURVES, II. *By* PROFESSOR SYAMA DAS MUKHOPADHYAYA 233
ON THE EQUATIONS OF MÖBIUS SURFACES OF ALL PITCHES. (PART II). *By* PROFESSOR C. E. CULLIS .. 245
THE THEORY OF ELECTRONS. *By* PROFESSOR E. P. HARRISON 265
THE FUTURE OF MATHEMATICS. *By* HENRI PONICARÉ 277
PROCEEDINGS 285
REVIEWS:
Introduction á la théorie des équations intégrales. *By* TRAJAN LALESCO 289
L'equation de Fredholm et ses applications á la Physique Mathematique. *By* HEYWOOD and FRÉCHET 289
Problémes et exercises de Mathématiques ginivales. *By* FABRY 290
NOTES AND NEWS 291
SOCIETIES AND ACADEMIES:
Academy of Sciences, Paris, October—December, 1909 .. 297
American Mathematical Society, October—December, 1909 298
London Mathematical Society, October—December, 1909 298

Calcutta Mathematical Society.

The third Monthly Meeting of the Society was held at the Senate House on the 23rd January, 1909.

The Hon'ble Mr. Justice Asutosh Mookerjee, M.A., D.Sc., D.L., President, was in the Chair.

The following candidates were proposed for election as members of the Society:—

Mrs. Snehalatā Maitra, B.A. (proposed by the Secretary); Mr. Abinash Chandra Ghosh, B.A., Lecturer, Bangabasi College, Calcutta (proposed by the Secretary); Mr. Baidya Nath Bose, M.A., Principal, D. J. College, Monghyr (proposed by the Secretary); Mr Nagendra Nath Banerjee, M.A., B.L., Pleader, Alipur (proposed by the Secretary); Mr. Jitendra Mohan Ray, C.E., Assistant Engineer, P.W.D., Rangoon (proposed by the Secretary).

Fifty-seven presents were announced as having been received, including, amongst others:—

Lamb, Hydrodynamics, 3rd ed.; Sir R. Ball, Theory of Screws; Whittaker, Modern Analysis; Campbell, Theory of Continuous Groups; Granville, Differential and Integral Calculus: (presented by the Hon'ble Dr. Asutosh Mookerjee). Cayley, Math. Papers, vol. i: (presented by Mr. Manindra Nath Bhuttacharjee). Lacroix, Calcul Différential et Intégral, 3 vols; Price, Infinitesimal Calculus, 2 vols; Arago, Popular Astronomy, 2 vols; Resal, Traité de Mécanique Générale, 7 vols; Biot, Traité d'Astronomie Physique, 4 vols: (presented by Dr. S. C. Bagchi).

The following papers were presented to the Society:—

Professor C. E. Cullis: On a certain Cubic Surface called the Möbius Surface.

Dr. S. C. Bagchi: On the Jordan Curves on Riemann's Surface.

Professor Syama Das Mookerjee: New Methods in the Geometry of a Plane Arc; Cyclic and Sextactic Points.

Mr. Phanindra Lal Gangooly: On a problem in Double-Star Astronomy.

Mr. P. N. Dutt: On Dynamical Deductions from certain equations.

The first two papers were partly read.

The February meeting of the Society was held at the Senate House on the 18th February, 1909.

The Hon'ble Mr. Justice Asutosh Mookerjee, M.A., D.Sc., D.L., President, was in the Chair.

The following candidates were proposed for election as members of the Society :—

Mr. Brojo Lal Mookerjee, M.A., Solicitor (proposed by the Secretary); Col. S. G. Burrard, F.R.S., R.E., Superintendent, Trigonometrical Surveys, Dehra Dun (proposed by the Secretary); Mr. J. deG. Hunter, B.A., (Cantab), Survey Department, Lahore (proposed by the Secretary); Mr. Mohini Mohan Ray, M.A., Professor of Mathematics, Krishnath College, Berhampur, Bengal (proposed by the Secretary); Mr. Indu Lal Bhuttacharjee, M.A.,B.L., Assistant Superintendent, Office of the Accountant General, Burma (proposed by the Secretary); Mr. J. C. Bagchi, B.A. (Cantab), Bar.-at-Law (proposed by the Secretary).

The following papers were presented to the Society :—

Dr. S. C. Bagchi: On Freedom and Geometry.

Mr. Phanindra Lal Gangooly: The Future of Mathematics; translated from an address by H. Poincaré read at the International Congress of Mathematicians held in Rome, 1908.

Professor Syama Das Mookerjee read his paper on New Methods in the Geometry of a Plane Arc, and Dr. Ganesh Prasad gave a brief account of his recent investigations into the solutions of partial differential equations of elliptic type.

The March meeting of the Society was held at the Senate House on the 17th March, 1909.

The Hon'ble Mr. Justice Asutosh Mookerjee, M.A., D.Sc., D.L, President, was in the Chair.

The following candidates were proposed for election as members of the Society :—

Mr. Lakshmi Narayan, M.A., Professor of Mathematics, Central Hindu College, Benares (proposed by Dr. Ganesh Prasad); Mr. Chandi Prasad, B.A., B.Sc., Demonstrator in Physics, Queen's College, Benares (proposed by Dr. Ganesh Prasad); Dr. G. Thibaut, C.I.E., Ph.D., D.Sc., Registrar, Calcutta University (proposed by the Secretary); Mr. Jnanendra Nath Gangooly, District Engineer, Howrah (proposed by the Secretary); Mr. Sudarsan Chakravarti, B.L., Vakil, Rajshahi (proposed by Mr. Bireswar Bagchi); Mr. Mukundo Nath Ghosh, B.L., Pleader, Rajshahi (proposed by Mr. Bireswar Bagchi); Mr. Kedareswar Acharya, M.B., Rajshahi (proposed by Mr Bireswar Bagchi); Mr. P. K. Dutt, M.A. (Cantab), Professor of Physics, Central Hindu College, Benares (proposed by Dr. Ganesh Prasad); Mr. Umesh Chandra Ghosh, M.A., Professor of Mathematics, Muir Central College, Allahabad (proposed by the Secretary); Mr. S. C. Ray, B.A., Bar.-at-Law (proposed by Mr. P. N. Dutt).

Mrs. Snehalata Maitra, Messrs. Abinash Chandra Ghosh, Baidya Nath Bose, Nagendra Nath Banerjee, Jitendra Mohan Ray were balloted for and duly elected members of the Society :—

The following papers were presented to the Society :—

Professor Hari Lal Chaudhuri: On Envelopes of Tangents to a System of Conics through four points (communicated by the President).

Dr. S. C. Bagchi: Sophus Lie and his Influence on Geometry.

Professor Syama Das Mookerjee: On Rates of Variation of the Osculating Conic.

Mr. Phanindra Lal Gangooly: On Infinite Determinants.

Mr. P. N. Dutt read his paper on Dynamical deductions from certain equations, and Dr. S. C. Bagchi gave an interesting account of the profound influence of Sophus Lie on Geometry.

BENEFACTORS OF THE SOCIETY.

Dr. Rash Behari Ghosh, Mr. R. N. Mookerjee, and Mr. Mahendra Nath Ray have each contributed two-hundred and fifty rupees to the General Funds of the Society.

The Method of Poincaré.*

The question is often asked, what are the mathematical works of Poincaré to which he owes his world-wide renown? Unfortunately, the fact that the name of a mathematician is universally known, does not render his theories of Mathematical Analysis more accessible. In these matters, the curiosity of the public is too modest and may be satisfied with words without any knowledge of the ideas. To satisfy the public curiosity in that way may be an amusing task, but it would be out of place here. The only advice which can be given to those who have a real desire to read and understand the mathematical works and memoirs of Poincaré, is that they should first prepare themselves by mathematical studies extending over a dozen years; three or four years would be sufficient for those who are already familiar with the elements taught in the lycées and écoles.

If we should give up our attempt to render accessible the results of the researches which have placed Poincaré at the side of the five or six great mathematicians of the nineteenth century and in the forefront of the mathematicians of the twentieth century, it need not perhaps be the same with the *method* employed in those researches. Here an endeavour will be made to describe briefly that method. The attempt is not indiscreet, for Poincaré has never sought to conceal the march of his genius.

The method of Poincaré is essentially active and constructive. He approaches a question, acquaints himself with its present condition without being much anxious about its history, finds out immediately the new analytical formulas by which the question can be advanced, deduces hastily the essential results, and then passes to another question. After having finished the writing of a memoir, he is sure to pause for a while, and think out how the exposition could be improved; but he would not, for a single instant, indulge in the idea of devoting several days to didactic work. Those days could be better utilised in exploring new regions.

All this is not specially applicable to mathematics. Let us examine more closely the mechanism made use of for discovery. The essential feature of that mechanism is, as we have already pointed out, the construction of new formulas. It is not useless that some stress is laid on this point, for this constructive power is the essential trait of the genius of Poincaré. The non-mathematical readers can be made to understand all this by means of a comparison. They know what arithmetical calculation is, and are often led to believe that mathematicians are in the habit of making interminable additions, multiplications, etc., and also extractions of cube roots. In reality, arithmetical operations are unique combinations of integral numbers formed of units which are all equal to one another. These operations can be compared to the construction of regular walls by means of bricks of a uniform size. Only the work requires some patience and a little care. On the contrary, analytical operations make use of extremely numerous materials and their variety is comparable to those of architectures where stone, marble, wood, iron, etc., are used. Those operations are as different from each other as cuirassé is from a Gothic Church. They have also with the architectural constructions this in common, that an impression of beauty is produced by the simplicity and elegance of the essential lines, without any effort by which the result has been obtained being apparent.

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* Translated from an article by Emile Borel which appeared in *La Revue du Mois*, March 10, 1909.

Poincaré is a great constructor. He can exactly adapt his construction to the end he desires to attain, without any difficulty being able to thwart his efforts. From this point of view, his activity can be compared to that of men of action who crush all the obstacles standing in their way. The difference is that the conquests of Poincaré are limited to the domain of thought.

It would be interesting to enquire to what extent this method of work has influenced the philosophy of Poincaré. Men of action generally look down with great contempt upon those individuals whom they bring into submission to their own will. So the man accustomed to see the formulas yielding to the force of his genius, may be tempted not to attribute to them a value absolutely superior to his individual judgment. If the philosophical conceptions of Poincaré are considered from this point of view, they will appear to the reader to be full of interest, and at the same time their true significance will be better understood.

The preceding lines may, indeed, be regarded as containing a sort of refutation, in the name of the science, of all positive and rational conception. But is it not contradictory to condemn in the name of science the foundation of all sciences? Recently in a mundane and academic ceremony, a historian boasted of never having understood the first principles of mathematics, and called those as *primaires*, who would not interpret the ideas of Poincaré in the same way as he did. He did but little reflect that in order to understand those ideas, he would require some knowledge of certain indispensable things, including perhaps that which is acquired in the course of primary education. Like that historian, certain persons refuse to admit that some knowledge of the elements of sciences is indispensable for being conversant with scientific philosophy. They prefer to treat with contempt those who attach any importance to the acquisition of those elements.

Let us now return to the mathematical method of Poincaré. That method can be best described by saying that Poincaré is more a conqueror than a coloniser. He boldly advances into unexplored regions, and then leaving to others the work of organisation, he proceeds towards other regions where his presence will be more necessary.

Poincaré attaches little importance to conceptions which cannot be immediately realised in a concrete form. This again is a trait of character which he has in common with men of action. He is neither a dreamer nor an ideologist, and it could almost be said, if there be no risk of a statement so paradoxical being misunderstood, that his brain works unremittingly and never pauses for reflection. Let us simply say that his method of work is too active to leave room for any reflection which does not immediately lead to a concrete result.

Thanks to that method, that he has been able to give to the world a scientific production which is the most considerable, since Gauss and Cauchy, and which, not ceasing to grow in extent every year, will perhaps end by constituting the most important contribution that a mathematician has ever made to the intellectual patrimony of humanity.

P. G.

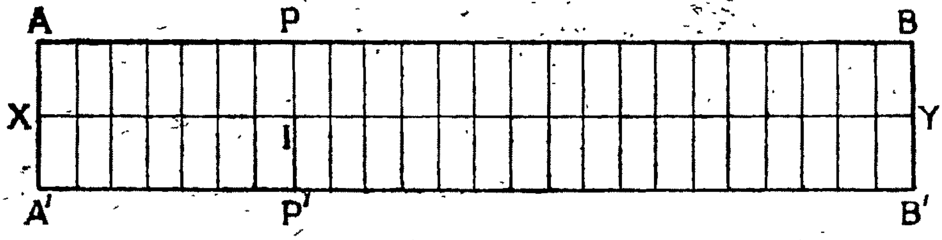
* * *

On a Certain Cubic Surface called the Möbius Surface.

(Part I.)

By C. E. Cullis.

1. Generation of the Surface.

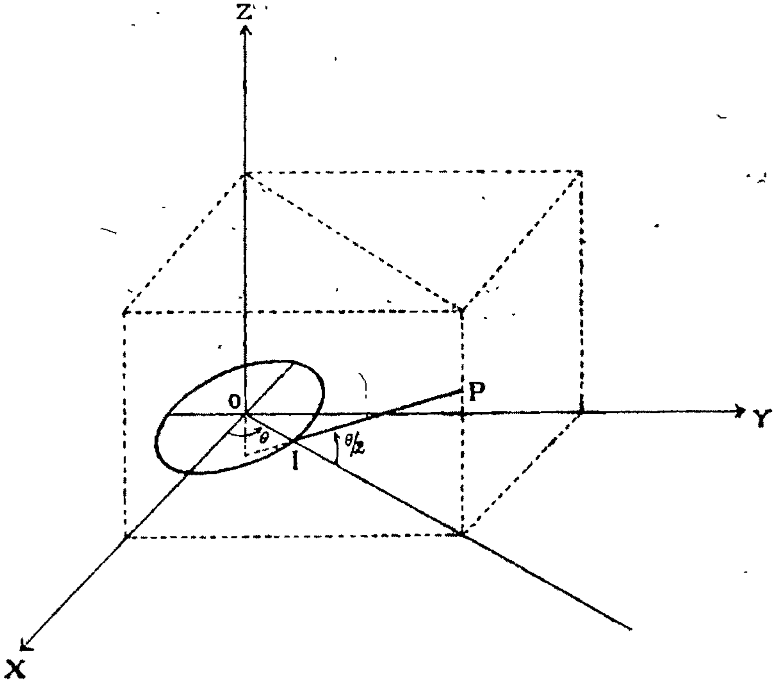


Let $AA'B'B$ be a rectangular strip of paper, and X and Y be the middle points of AA' and BB'. By giving a *curl* to this strip and bringing the sides BB', AA' into coincidence with one another, a circular cylindrical surface can be formed. The points B, B' will then coincide respectively with the points A, A', and Y will coincide with X. If, before curling the strip, we give it a ***half-twist***, about XY as axis through an angle of 180°, and then bring the sides AA', BB' into coincidence, B coinciding with A' and B' with A, we get an interesting surface which may be called a *Möbius Ring*. It is a finite surface possessing only one side, and bounded by a single closed curve. When it is cut along the closed curve formed by XY, it does not fall into two pieces, but remains a single connected surface.

If we imagine the strip to be of elastic material, the curling and twisting might proceed at uniform rates in such a manner that straight lines parallel to AA' and BB' remain straight lines. If further the sides AA', BB' be supposed to be of infinite length, the surface thus obtained is a ruled surface which cuts itself and is generated by the straight lines whose equations in terms of a parametrical angle θ are

$$\frac{x-a\cos\theta}{\cos\theta\cos\frac{\theta}{2}}=\frac{y-a\sin\theta}{\sin\theta\cos\frac{\theta}{2}}=\frac{z}{\sin\frac{\theta}{2}}.$$

Each generator intersects the circle $z=0$, $x^2+y^2=a^2$ at right angles, and also intersects the z-axis. The angle θ is that which the projection of the generator on the xy-plane makes with the x-axis, and the generator itself makes an angle $\frac{\theta}{2}$ with the xy-plane.



This ruled surface is the surface whose properties will now be considered, and to which the name of *Möbius Surface*[*] is here applied. It may be described as the surface generated by a straight line IP which intersects a fixed circle orthogonally at I, and turns about the tangent line to the circle at I with one-half the angular velocity with which I describes the circle.

Returning to the finite rectangular strip, we may, before bringing the sides AA', BB' into coincidence, give to the strip *any number* of curls and *any number* of half-twists. According as the number of half-twists is even or odd, the points B, B' will then be made to coincide respectively with A, A' or with A', A. The corresponding ruled surfaces are those whose generators are

---

[*] The name *Möbius Surface* is given to the *finite* surface first described in a book by *C. Neumann* on *Riemann's Theorie der Abel'schen Integrale*, where it is stated that the properties of this surface were first considered by *Möbius*. The name is here applied to a definite ruled surface, the less definite finite surface being called a *Mobius Ring*.

$$\frac{x-a\cos\theta}{\cos\theta\cos\frac{m}{2n}\theta}=\frac{y-a\sin\theta}{\sin\theta\cos\frac{m}{2n}\theta}=\frac{z}{\sin\frac{m}{2n}\theta},$$

where m and n are integers which are prime to one another. It may be considered that m is the number of half-twists through 180°, and n the number of curls. To obtain all the generators, θ must be made to assume all values from 0 to $2n\pi$. All these surfaces might be called Möbius surfaces, and the number $\frac{m}{2n}$, which is the ratio of the rate of twisting to the rate of curling, might be called the *pitch* of the corresponding surface. Then the surface now to be considered would be that special Möbius surface whose pitch is $\frac{1}{2}$.

2. First Form of the Equation of the Surface—$\phi(x,y,z)=0$.

The equations of any generator can be written as

$$\left.\begin{aligned} x&=\cos\theta\left(a+z\cot\frac{\theta}{2}\right)\\ y&=\sin\theta\left(a+z\cot\frac{\theta}{2}\right)\end{aligned}\right\};\quad\text{or}\quad\left.\begin{aligned} y&=x\tan\theta\\ y&=\sin\theta\left(a+z\cot\frac{\theta}{2}\right).\end{aligned}\right\}$$

If t be put for $\tan\frac{\theta}{2}$, these become

$$y=\frac{2xt}{1-t^2},\quad y=\frac{2(z+at)}{1+t^2};$$

or

$$\left.\begin{aligned} y&=z+(x+a)t\\ y\,(1-t^2)&=2xt\end{aligned}\right\}.$$

Eliminating t, the equation of the surface is obtained in the form

$$\phi\,(x,\,y,\,z)\equiv y\,(x^2+y^2+z^2-a^2)-2z\,(x^2+y^2+ax)=0.$$

3. Singular Points. The Double Line.

Equating

$$\frac{\delta\phi}{\delta x},\ \frac{\delta\phi}{\delta y}\text{ and }\frac{\delta\phi}{\delta z}$$

to zero, it will be seen that every point on the straight line

$$x+a=0,\ y-z=0,$$

is a singular point, and it is easily seen that at every one of these points there are two tangent planes. This line is then a *double line* along which the surface intersects itself.

Every plane section of the surface is a cubic curve having a node on the double line, and a cubic curve cannot have more than one singular point. Hence the surface can have no other singular points besides those forming the double line.

4. The Direct and Oblique Axes of the Surface.

The straight lines $x=0$, $y=0$ and $x+a=0$, $y-z=0$ are intersected by every generator. They lie on the surface but are not included amongst the generators. They will be called respectively the *direct axis* and the *oblique axis* of the surface. The generator θ cuts the direct axis where

$$z = -a \tan\frac{\theta}{2},$$

and the oblique axis where $z = -a\tan\theta$. Of course *two* generators pass through every point of the oblique axis (which is the double line), whilst only one generator passes through each point of the direct axis.

5. Second Form of the Equation of the Surface—$\psi(x, y, z)=0$.

If the origin is transferred to the point $x=-a$, $y=0$, $z=0$, and the axes of y and z are then turned through an angle of 45° about the x-axis, so that the *oblique axis becomes the new z-axis*, the equation of the surface referred to the new axes is

$$\psi(x, y, z) \equiv x^3(3y-z)+2y^3(y+z)-4axy=0.$$

Equating

$$\frac{\delta\psi}{\delta x},\ \frac{\delta\psi}{\delta y} \text{ and } \frac{\delta\psi}{\delta z}$$

to zero, it is easily seen *analytically* that there are no singular points except those on the z-axis.

The straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z-z'}{n}$$

drawn through any point $(0, 0, z')$ on the z-axis (or double line), cuts the surface in three consecutive points if it lies in one of the planes

$$z'x^2+4axy-2z'y^2=0.$$

These are the two tangent planes to the surface at the point $(0, 0, z')$.

6. Sections of the Surface $\phi(x, y, z)=0$ by Planes Parallel to the Co-ordinate Planes.

The expression $\phi(x, y, z)$ is given in § 2.

(1) *Section by* $x=k$.

This is the curve

$$y(y-z)^2+(k+a)\,[(k-a)y-2kz]=0.$$

There is no finite double point, the node being at infinity.

There are three finite asymptotes, *viz.*,

$$y=0,\ y-z=\pm(a+k).$$

The curve is symmetrical about the origin, which is a centre.

The origin is also a point of inflexion, the tangent there being

$$z = \frac{k-a}{2k}y.$$

The generator θ meets the section where

$$y = k \tan \theta$$

$$z = (k \sec \theta - a) \tan \frac{\theta}{2}.$$

When $k = 0$, the curve degenerates into the three straight lines

$$y(y-z+a)\,(y-z-a) = 0.$$

When $k = -a$, it degenerates into the three straight lines

$$y\,(y-z)^2 = 0.$$

The forms of the curve for the values

$$4a, \frac{a}{2}, -\frac{a}{2}, -\frac{3a}{2}$$

of k are shown in Plates I and II.

It consists of three separate infinite branches. Two of them are hyperbolic in character. The third lies between these two, and is serpentine in character, having a point of inflexion at the origin.

(2) *Section by* $y = k$.

This is the curve

$$2x^2z - [k(x^2+z^2) - 2axz] + 2k^2z + k(a^2 - k^2) = 0.$$

It has one double point, (a node), whose co-ordinates are $x = -a$. $z = k$.

There is one finite asymptote $z = \frac{k}{2}$.

The other asymptotes are $x^2 = \infty$, forming a double line at infinity.

If the origin were transferred to the double point, the equation of the curve would become

$$2x^2z + [kx^2 - 2axz - kz^2] = 0,$$

and the tangents at the origin would be

$$kx^2 - 2axz - kz^2 = 0.$$

Thus the tangents at the double point are real and at right angles to one another.

The generator θ meets the section where

$$x = k \cot \theta$$

$$z = (k \operatorname{cosec} \theta - a) \tan \frac{\theta}{2}.$$

When $k = 0$, the curve degenerates into the three straight lines

$$xz(x + a) = 0.$$

The form of the curve for the cases $k=4a$, $k=\frac{a}{2}$ is shown in Plate III.

It consists of two infinite branches which intersect and so form a node and a loop, which extends to infinity and is parabolic in character.

(3) *Section by $z=k$.*

This is the curve

$$y\,(x^2+y^2)-2k(x^2+y^2)+[(k^2-a^2)y-2kax]=0.$$

It has one double point, a node, whose co-ordinates are $x=a$, $y=k$.

There is only one real asymptote, *viz.*, $y=2k$.

If the origin were transferred to the double point, the equation of the curve would become

$$y\,(x^2+y^2)+[ky^2-2axy-kx^2]=0.$$

Then x would be real only when $y^2<a^2+k^2$.

Thus the tangents at the origin are real and at right angles to one another; also the curve lies entirely between the two straight lines

$$y=k\pm\sqrt{a^2+k^2},$$

so that $k\pm\sqrt{a^2+k^2}$ are the maximum and minimum values of y.

The generator θ meets the section where

$$x=\cos\theta\left(a+k\cot\frac{\theta}{2}\right)$$

$$y=\sin\theta\left(a+k\cot\frac{\theta}{2}\right).$$

When $k=0$ the curve degenerates into the straight line $y=0$ and the circle $x^2+y^2-a^2=0$.

The forms of the curve for the values $4a, \frac{a}{2}, 0, -\frac{a}{2}, -4a$ of k are shown in Plate IV.

It consists of a single infinite branch which intersects itself, forming a node and a finite loop.

7. Sections of the Surface $\psi\,(x, y, z)=0$ by Planes Parallel to the Co-ordinate Planes.

The expression $\psi\,(x, y, z)$ is given in § 5.

(1) *Section by $x=k$.*

This is the curve

$$2y^2(y+z)+k\,[(3k-4a)y-kz]=0.$$

It is the section of $\phi\,(x, y, z)=0$ by $x=k-a$, referred to axes turned through 45°.

The three finite asymptotes are

$$y+z=0,\ y=\pm\frac{k}{\sqrt{2}}.$$

The origin is a centre and a point of inflexion, the equation of the tangent there being

$$z=\frac{3k-4a}{k}\,y.$$

The generator θ meets the section where

$$\sqrt{2}z=2(k-a)\tan\theta-k\tan\frac{\theta}{2}.$$

$$\sqrt{2}y=k\tan\frac{\theta}{2}.$$

The forms of the curve for the cases

$$k=4a,\ k=\frac{a}{2}$$

are shown in Plate VI.

(2) *Section by $y=k$.*

This is the curve

$$x^2z-3kx^2+2k(2ax-kz)-2k^3=0.$$

There is no finite double point, the node being at infinity.

There are three finite asymptotes, *viz.*

$$y=3k,\ x=\pm\sqrt{2}\,k.$$

The co-ordinate z is real for all values of x.

When $k^2<\frac{a^2}{2}$, x is real for all values of z.

When $k^2>\frac{a^2}{2}$, x is real if z does not lie between $k\pm\sqrt{4k^2-a^2}$.

In the latter case z has real maximum and minimum values

$$k+\sqrt{4k^2-2a^2},\ k-\sqrt{4k^2-2a^2},$$

which occur when x has the values

$$\frac{k}{a}\left[2k+\sqrt{4k^2-2a^2}\right],\ \frac{k}{a}\left[2k-\sqrt{4k^2-2a^2}\right].$$

The generator θ cuts the section where

$$x=\sqrt{2}k\ \cot\frac{\theta}{2}$$

$$z=-k-\left[\sqrt{2}a-2k\ \cot\frac{\theta}{2}\right]\tan\theta=k\,(2\sec\theta+1)-\sqrt{2}a\tan\theta.$$

When $k=0$, the curve degenerates into the two straight lines $x^2z=0$.

When $k = \pm\dfrac{a}{\sqrt{2}}$, it degenerates into the straight line $x = a$ and the hyperbola

$$(x+a)\left(z \mp \frac{3a}{\sqrt{2}}\right) + 2\sqrt{2}\,a^2 = 0.$$

The forms of the curve when k has the values $4a$, $\dfrac{3a}{4}$, $\dfrac{a}{\sqrt{2}}$, $\dfrac{a}{2}$ are shown in Plates VII and VIII.

When $k > \dfrac{a}{\sqrt{2}}$ or $< -\dfrac{a}{\sqrt{2}}$, it consists of three separate infinite branches, hyperbolic in character, two of them having perpendicular asymptotes, and the third having parallel asymptotes. When $k < \dfrac{a}{\sqrt{2}}$ and $> -\dfrac{a}{\sqrt{2}}$, it consists of two infinite branches, hyperbolic in character, and one serpentine branch.

(3) *Section by* $z = k$.

This is the curve

$$y\,(3x^2 + 2y^2) - [k(x^2 - 2y^2) + 4axy] = 0.$$

The origin is a double point, (a node), the tangents there being

$$\frac{y}{x} = \frac{2a \pm \sqrt{4a^2 + 2k^2}}{2k}.$$

There is only one real asymptote, *viz.* $y = \dfrac{k}{3}$.

The points where y is a maximum or minimum are given by

$$3y^2 + 2ky - (k^2 + 2a^2) = 0, \qquad x\,(k - 3y) + 2ay = 0.$$

The co-ordinate x is real when y lies between the roots of the first of these equations. Thus the curve lies entirely between the two lines

$$y = \frac{-k \pm \sqrt{4k^2 + 8a^2}}{3}.$$

The generator θ cuts the curve where

$$x = \frac{\sqrt{2}k + 2a\tan\theta}{2\tan\theta - \tan\frac{1}{2}\theta},$$

$$y = x\left[\frac{1}{\sqrt{2}}\tan\frac{\theta}{2}\right] = \frac{k + \sqrt{2}a\tan\theta}{2\sec\theta + 1}.$$

When $k = 0$, the curve degenerates into the straight line $y = 0$ and the ellipse $3x^2 + 2y^2 - 4ax = 0$.

The forms of the curve for the values $4a$, $\dfrac{a}{2}$, 0, $-\dfrac{a}{2}$, $-4a$ of k are shown in Plate IX.

It consists of a single infinite branch which intersects itself so as to form a node and a finite loop.

8. Plane Sections which are Conics. Sections by Planes through the Direct Axis, the Oblique Axis and the Initial Generator.

The section made by any plane through any generator consists of that generator and a conic.

Any plane through the direct axis cuts the surface in the direct axis and two generators.

Any plane through the oblique axis or double line cuts the surface in that double line and one generator.

The tangent plane at any point contains the generator through that point, and consequently cuts the surface in a conic.

If the equation of the surface is taken in the form $\phi(x, y, z)=0$, then any plane through the initial generator, (for which $\theta=0$), is of the form $z=y \tan \alpha$. To obtain the corresponding section, the axes will first be changed to a new set by the transformation scheme

	x'	y'	z'
x	1	0	0
y	0	$\cos \alpha$	$-\sin \alpha$
z	0	$\sin \alpha$	$\cos \alpha$

and then in the transformed equation z will be put equal to zero.

The transformed equation of the surface is

$$(y \cos \alpha - z \sin \alpha)(x^2+y^2+z^2-a^2)$$
$$-2(y \sin \alpha + z \cos \alpha)(x^2+y^2 \cos^2\alpha + z^2 \sin^2\alpha - 2yz \cos \alpha \sin \alpha + ax)=0.$$

Hence the equation of the section is, besides $y=0$,

$$x^2(1-2t)+y^2\,\frac{(1-t)^2}{1+t^2}-2axt-a^2=0, \text{ where } t=\tan \alpha.$$

This curve is a conic whose centre lies on the x-axis where

$$x=\frac{at}{1-2t}.$$

It is an ellipse if $t<\frac{1}{2}$, and a hyperbola if $t>\frac{1}{2}$.

If $t=\frac{1}{2}$, it is the parabola $y^2=5a(x+a)$.

If $t=1$, it is the double line $(x+a)^2=0$.

If $t=\infty$, it is the pair of parallel lines $x(x+a)=0$.

9. Construction of the Surface.

Every generator will cut two of the faces of the cube whose sides are $x = \pm k$, $y = \pm k$, $z = \pm k$. If then such a cube be formed and the points be determined in which the generators cut the faces, then a model of the surface can be constructed by joining every pair of corresponding points by strings. In Table I. (see pp. 20 and 21) the co-ordinates of these points are given, when $k = 4a$, for the generators corresponding to the values $0, 10°, 20°, \ldots 350°$ of θ, the first form of the equation of the surface being used. The last two entries refer to the direct and oblique axes. In Table II. (see pp. 22 and 23) the corresponding co-ordinates are given for the case $k = 5a$, when the second form of the equation of the surface is used. In the latter case the faces $y = \pm 5a$ can be dispensed with if the faces $z = \pm 5a$ are slightly extended, as will be made clear by an inspection of Plates V and X.

The surface may be described as consisting of a spiral sheet and a double funnel which merge into one another along those two generators, 90° and 270°, which are parallel to the oblique axis, in such a manner as to divide the whole of space into two regions. The double funnel is formed by the generators between 90° and 270°, and the spiral sheet of the remaining generators. Each region of space is bounded by one side of the spiral sheet, the outside of one funnel, and the inside of the other funnel. Every point of the surface can be described by two points starting in juxtaposition to one another on opposite sides of the surface. Each of these points can roam unhindered over one side of the spiral sheet, the outside of one funnel, and the inside of the other funnel.

Sections by planes cutting through the double funnel are shown in the upper figure of Plate II and the lower figures of Plates VI and VIII.

10. Generators Parallel to a given Plane.

The generator θ is parallel to the plane $lx + my + nz = 0$

if
$$l \cos\theta + m \sin\theta + n \tan\frac{\theta}{2} = 0.$$

Let $t = \tan\dfrac{\theta}{2}$. Then this condition is

$$l(1 - t^2) + 2mt + nt(1 + t^2) = 0$$

or
$$nt^3 - lt^2 + (2m + n)t + l = 0. \qquad (1)$$

Since this is a cubic equation, the number of generators parallel to a given plane is either 1 or 3. The asymptotes of the section of the surface by the plane are parallel to these generators. Hence they are at the same time either 1 or 3 in number; also *sections by parallel planes have parallel asymptotes.*

A plane which is parallel to two generators θ_1, θ_2 must also be parallel to a third generator, and equation (1) shows that these three generators parallel to the same plane are connected by the relation

$$t_1t_2t_3 + t_1 + t_2 + t_3 = 0 \quad \text{or} \quad t_3 = -\frac{t_1+t_2}{1+t_1t_2},$$

where $t_1 = \tan\frac{1}{2}\theta_1$, $t_2 = \tan\frac{1}{2}\theta_2$, $t_3 = \tan\frac{1}{2}\theta_3$.

The fact that planes whose sections have three real asymptotes are identical with the planes which are parallel to two generators, enables us to find the general equation of such planes. Let (λ, μ, ν) be the direction cosines of the normal to a plane which is parallel to the two generators θ_1, θ_2. Then

$$\lambda(1-t_1^2) + 2\mu t_1 + \nu(t_1+t_1^3) = 0$$
$$\lambda(1-t_2^2) + 2\mu t_2 + \nu(t_2+t_2^3) = 0$$
$$\therefore \quad \frac{\lambda}{2t_1t_2(t_1+t_2)} = \frac{\mu}{t_1^2t_2^2-(t_1+t_2)^2-1} = \frac{\nu}{2(t_1t_2+1)}.$$

Hence the general equation of a plane whose section has three real asymptotes is

$$2t_1t_2(t_1+t_2)x + [t_1^2t_2^2-(t_1+t_2)^2-1]y + 2(t_1t_2+1)z + c = 0,$$

where t_1, t_2, c may assume independently all real values from $-\infty$ to $+\infty$.

Put $t_1t_2 = a$, $t_1+t_2 = b$.

Then the general equation of such a plane is

$$2abx + (a^2-b^2-1)y + 2(a+1)z + c = 0 \qquad (2)$$

where a, b, c may receive all real values such that $b^2 > 4a$.

Since two generators always intersect the direct axis in two different points, a plane which *contains* two generators must also contain the direct axis, and theerfore be a meridian plane. Now every meridian plane contains two and only two generators. These are generators at opposite extremities of a diameter of the guiding circle, or, what is the same thing, generators which meet on the oblique axis. Hence two generators can only be *contained* in the same plane, when they meet on the oblique axis and pass through opposite extremities of a diameter of the guiding circle. In this case the third generator parallel to the plane is the generator, for which $\theta = 180°$, which is parallel to the direct axis.

If $f(t)$ be written for the left-hand side of equation (1), we have

$$f(1) = 2\,(m+n), \quad f(-1) = -2\,(m+n).$$

Hence the equation always has *one* real root lying between -1 and $+1$.

Accordingly one of the generators for which θ lies between 0° and 90°, or between 270° and 360°, is parallel to every plane. That is, the spiral sheet joining the generators 90° and 270° contains at least one generator parallel to every plane.

Table I.—Intersections of the generators of $\phi=0$ with the faces of the cube formed by the planes $x=\pm 4a$, $y=\pm 4a$, $z=\pm 4a$.

	$x=4a$		$y=4a$		$z=4a$		$x=-4a$		$y=-4a$		$z=-4a$	
$\theta°$	$\frac{y}{a}$	$\frac{z}{a}$	$\frac{z}{a}$	$\frac{x}{a}$	$\frac{x}{a}$	$\frac{y}{a}$	$\frac{y}{a}$	$\frac{z}{a}$	$\frac{z}{a}$	$\frac{x}{a}$	$\frac{x}{a}$	$\frac{y}{a}$
0	0	0					0	0				
10	·705	·268					−·705	−·443				
20	1·456	·574					−1·456	−·927				
30	2·309	·970					−2·309	−1·506				
40	3·356	1·536					−3·356	−2·264				
50			1·969	3·356					−2·901	−3·356		
60			2·089	2·309					−3·244	−2·309		
70			2·280	1·456					−3·681	−1·456		
80			2·570	·705							−·655	−3·713
90			3	0							0	−3
100			3·649	−·705							·410	−2·324
110					−1·300	3·571					·616	−1·691
120					−1·655	2·867					·655	−1·184
130					−1·838	2·191					·558	−·659
140					−1·884	1·581					·352	−·296
150					−1·798	1·035					·061	−·035
160					−1·597	·581					−·282	·108
170					−1·330	·234					−·640	·113

180					−1	0					−1	0
190					−·640	−·113					−1·330	−·234
200					−·282	−·103					−1·597	−·581
210					·061	·035					−1·793	−1·035
220					·452	·296					−1·884	−1·581
230					·552	·659					−1·838	−2·191
240					·655	1·134					−1·655	−2·867
250					·616	·691					−1·300	−3·571
260					·410	2·324			−3·649	−·705		
270					0	3			−3	0		
280					−·655	3·713			−2·569	·705		
290			3·681	−1·456					−2·280	1·456		
300			3·244	−2·309					−2·089	2·309		
310			2·901	−3·356					−1·969	3·356		
320	−3·356	−1·536					3·356	2·264				
330	−2·309	−·970					2·309	1·506				
340	−1·456	−·574					1·456	·927				
350	−·705	−·268					·705	·443				
360	0	0					0	0				
dir.					0	0					0	0
obl.			4	−1	−1	4			−4	−1	−1	−4

able II.—Intersections of the generators of $\psi = 0$ with the faces of the cube formed by the planes $x = \pm 5a$, $y = \pm 5a$, $z = \pm 5a$.

	$x = 5a$		$y = 5a$		$z = 5a$		$x = -5a$		$y = -5a$		$z = -5a$	
θ°	$\frac{y}{a}$	$\frac{z}{a}$	$\frac{z}{a}$	$\frac{x}{a}$	$\frac{x}{a}$	$\frac{y}{a}$	$\frac{y}{a}$	$\frac{z}{a}$	$\frac{z}{a}$	$\frac{x}{a}$	$\frac{x}{a}$	$\frac{y}{a}$
0	0	0					0	0				
10	·308	·688					−·308	−1·187				
20	·623	1·432					−·623	−2·461				
30	·945	2·315					−·945	−3·948				
40	1·285	3·456									−4·103	−1·056
50					4·931	1·626					−2·445	−·806
60					3·649	1·490					−1·250	−·510
70					2·621	1·298					−·329	−·168
80					1·753	1·040					·407	·241
90					1	·707					1	·707
100					·341	·287					1·469	1·238
110					−·228	−·230					1·815	1·833
120					−·694	−·850					2·027	2·488
130					−1·035	−1·570					2·088	3·166
140					−1·218	−2·367					1·977	3·841
150					−1·211	−3·195					1·683	4·442
160					−·991	−3·975					1·219	4·887
170			−4·905	·618	−·570	−4·608					·630	5·092

180			−5	0	0	−5			5		0	5
190					·680	−5·092			4·905	·618	−·570	4·608
200					1·219	−4·887					−·991	3·975
210					1·683	−4·442					−1·211	3·195
220					1·977	−3·841					−1·218	2·367
230					2·088	−3·166					−1·035	1·570
240					2·027	−2·483					−·694	·850
250					1·815	−1·833					−·228	·230
260					1·469	−1·238					·341	−·287
270					1	−·707					1	−·707
280					·407	−·241					1·753	−1·040
290					−·329	·163					2·621	−1·298
300					−1·249	·510					3·649	−1·490
310					−2·445	·806					4·931	−1·626
320	−1·285	−3·456			−4·108	1·056						
330	−·945	−2·315					·945	3·948				
340	−·623	−1·432					·623	2·461				
350	−·308	−·688					·308	1·187				
360	0	0					0	0				
dir.			−5	1	1	−5			5	1	1	5
obl.					0	0					0	0

The condition that equation (1) may have *three* real roots is

$$l^2[l^2+m^2-11n^2-8mn] \;>\; n(n+2m)^2$$

To find the generators perpendicular to a given generator a, we must put

$$l : m : n \;=\; \cos a : \sin a : \tan \frac{a}{2},$$

thus obtaining

$$\cos(\theta-a) \;+\; \tan\frac{a}{2}\tan\frac{\theta}{2} = 0.$$

This equation can be solved graphically by finding the points of intersection of the curves

$$y=\cos(x-a),\; y \;=\; -\tan\frac{a}{2}\tan\frac{x}{2}\;.$$

A section becomes degenerate when its plane passes through one of the generators to which it is parallel. Thus there are three degenerate forms of the section made by a plane parallel to two, and therefore to three given generators.

11. Section of $\phi(x, y, z)=0$ by *any* Plane.

Let the plane be $lx+my+nz-p=0$, where (l, m, n) are direction-cosines. It cuts the direct axis in the point

$$\left(0, 0, \frac{p}{n}\right).$$

If we take this point as a new origin, the normal to the plane as a new z-axis, and two mutually perpendicular lines in the plane with direction-cosines (l_1, m_1, n_1), (l_2, m_2, n_2) as new axes of x and y, we can obtain the equation of the section by transforming the equation

$$y\,(x^2+y^2+z^2-a^2)-2z(x^2+y^2+ax) \;=\; 0$$

by the scheme

	x'	y'	z'
x	l_1	l_2	l
y	m_1	m_2	m
$z-\frac{n}{p}$	n_1	n_2	n

and putting $z=0$ in the result.

The transformed equation of the surface is

$$(m_1x+m_2y+mz)\left[x^2+y^2+z^2+\frac{2p}{n}\,(n_1x+n_2y+nz)\;+\frac{p^2}{n^2}-\;a^2\right]$$

$$-2\left(n_1x+n_2y+nz+\frac{p}{n}\right)\times$$

$$[(l_1x+l_2y+lz)^2+\;(m_1x+m_2y+mz)^2+a\;(l_1x+l_2y+lz)]=0\;;$$

and therefore the equation of the section is

$$(m_1x+m_2y)\left[x^2+y^2+\frac{2p}{n}(n_1x+n_2y)+\frac{p^2}{n^2}-a^2\right] \qquad (A.)$$

$$-2\left(n_1x+n_2y+\frac{p}{n}\right)\left[(l_1x+l_2y)^2+(m_1x+m_2y)^2+a(l_1x+l_2y)\right]=0.$$

Since the terms of the third degree do not involve p, it follows that *sections by parallel planes have parallel asymptotes.*

The co-ordinates of the double point are

$$x=\frac{m_1+n_1}{m+n}(p+al)-\left(al_1+\frac{n_1}{n}p\right)$$

$$y=\frac{m_2+n_2}{m+n}(p+al)-\left(al_2+\frac{n_2}{n}p\right).$$

If the new origin is taken at the foot of the perpendicular drawn to the plane from the centre of the guiding circle, the transformation scheme is

	x'	y'	$z'+p$
x	l_1	l_2	l
y	m_1	m_2	m
z	n_1	n_2	n

and the equation of the section is

$$(m_1x+m_2y+mp)\left[x^2+y^2+p^2-a^2\right]-2(n_1x+n_2y+np)$$
$$\times\left[(l_1x+l_2y+lp)^2+(m_1x+m_2y+mp)^2+a(l_1x+l_2y+lp)\right]=0 \qquad (B.)$$

The co-ordinates of the double point are then

$$x=\frac{m_1+n_1}{m+n}(p+al)-al_1$$

$$y=\frac{m_2+n_2}{m+n}(p+al)-al_2.$$

If in either case the origin is transferred to the double point, the equation of the section takes the form

$$[m_1-2n_1(l_1^2+m_1^2)]x^3+[m_2-4n_1(l_1l_2+m_1m_2)-2n_2(l_1^2+m_1^2)]x^2y$$
$$+[m_1-4n_2(l_1l_2+m_1m_2)-2n_1(l_2^2+m_2^2)]xy^2+[m_2-2n_2(l_2^2+m_2^2)]y^3$$
$$+\frac{p+al}{m+n}\left\{\left[(m_1x+m_2y)-(n_1x+n_2y)-\frac{(m+n)a}{p+al}(l_1x+l_2y)\right]^2\right.$$
$$\left.-\left(1+\frac{(m+n)^2a^2}{(p+al)^2}\right)(l_1x+l_2y)^2\right\}=0. \qquad (C.)$$

This form shows the existence of the two real tangents at the double point.

The equation of the section can in each case be simplified by a suitable choice of the axes of x and y in its plane.

If we choose the *x-axis* to be *perpendicular to the direct axis* of the surface, we have $n_1 = 0$ and $l_1 l + m_1 m = 0$; also (l_2, m_2, n_2) is perpendicular to both (l_1, m_1, n_1) and (l, m, n);

$$\therefore\quad l_1 = -\frac{m}{\nu},\ m_1 = \frac{l}{\nu},\ n_1 = 0;$$

$$l_2 = -\frac{nl}{\nu},\ m_2 = -\frac{mn}{\nu},\ n_2 = \nu;\ \text{where } \nu = \sqrt{l^2+m^2} = \sqrt{1-n^2}\cdot$$

Equation A then becomes

$$\begin{aligned} n^2[lx^3 - (mn+2\nu^2)x^2y + lxy^2 - (mn+2n^2\nu^2)y^3] \\ -2n\nu[px^2 - (lp+mna)xy + (\overline{m+n}\,p - nla)ny^2] \\ +[\,(lp^2+2mnap - ln^2a^2)\,x - (mp^2 - 2nlap - mn^2a^2)ny\,] = 0, \end{aligned} \qquad A_1$$

the co-ordinates of the double point being

$$x = \frac{(l^2+m^2+mn)a + lp}{\nu(m+n)},\ y = \frac{nla - mp}{n\nu(m+n)}.$$

Equation B becomes

$$\begin{aligned} (lx - mny + \nu mp)(x^2+y^2+p^2-a^2) - 2(\nu y + np)[\nu(x^2+n^2y^2) \\ - max - n(la+2\nu^2p)y + \nu^2p^2 + \nu lap] = 0, \end{aligned} \qquad B_1$$

the co-ordinates of the double point being

$$x = \frac{(\nu^2+mn)a + lp}{\nu(m+n)},\ y = \frac{(\nu^2-mn)p + la}{\nu(m+n)}.$$

In this case the generator θ meets the section in the point given by

$$\nu x = \frac{(l\sin\theta - m\cos\theta)\left[na\sin\frac{\theta}{2} + p\cos\frac{\theta}{2}\right]}{(l\cos\theta + m\sin\theta)\cos\frac{\theta}{2} + n\sin\frac{\theta}{2}},$$

$$\nu y = \frac{-(l\cos\theta + m\sin\theta)\left[a\sin\frac{\theta}{2} + np\cos\frac{\theta}{2}\right] + \nu^2 p\sin\frac{\theta}{2}}{(l\cos\theta + m\sin\theta)\cos\frac{\theta}{2} + n\sin\frac{\theta}{2}};$$

and this last result may be used to plot the section made by any plane whatever.

Equation C under the same circumstances becomes

$$\begin{aligned} lx^3 - (mn+2\nu^2)x^2y + lxy^2 - (mn+2n^2\nu^2)y^3 \\ + \frac{(p+al)}{\nu(m+n)}\left\{\left[lx - (mn+\nu^2)y + \frac{(m+n)a}{p+al}(mx+ly)\right]^2 - \left[1 + \frac{(m+n)^2a^2}{(p+al)^2}\right]\right. \\ \left.\times\left[mx + nly\right]^2\right\} = 0. \end{aligned}$$

If we choose the *x-axis* to be *perpendicular to the initial generator*, $\theta=0$, we have $l_1=0$, $m_1m+n_1n=0$; and therefore

$$\left.\begin{aligned} l_1 &= 0, \quad m_1 = \frac{n}{\lambda}, \quad n_1 = -\frac{m}{\lambda}, \\ l_2 &= -\lambda, \quad m_2 = \frac{lm}{\lambda}, \quad n_2 = \frac{ln}{\lambda}, \end{aligned}\right\}\text{where } \lambda = \sqrt{m^2+n^2} = \sqrt{1-l^2}\cdot$$

Equation C then becomes

$$\begin{aligned} & n(\lambda^2+2mn)x^3 + l(m+n)[m(m+n)+2n(m-n)]x^2y \\ & +[(n+2m)\lambda^2 - 6\,l^2mn^2]xy^2 + l[(m-2n)\lambda^2+2l^2n^2]y^3 \qquad C_2 \\ & +\frac{p+al}{m+n}\lambda\left\{\left[(m+n)x+(m-n)ly+\frac{(m+n)a}{p+al}\lambda^2y\right]^2\right. \\ & \qquad \left. -\left[1+\frac{(m+n)^2a^2}{(p+al^2)}\right]\lambda^4y^2\right\}=0. \end{aligned}$$

12. Section of $\psi(x, y, z)=0$ by *any* Plane.

Let the plane be $lx+my+nz-p=0$. If we take the point where the plane cuts the oblique axis, *i.e.*, the double point, as origin, and take two mutually perpendicular lines in the plane with direction-cosines (l_1, m_1, n_1) (l_2, m_2, n_2) as new axes of x, and y, the transformation scheme is

	x'	y'	z'
x	l_1	l_2	l
y	m_1	m_2	m
$z-\frac{p}{n}$	n_1	n_2	n

and the equation of the section referred to the new axes of x and y in its plane is

$$\begin{aligned} & (l_1x+l_2y)^2\left[(3m_1-n_1)x+(3m_2-n_2)y-\frac{p}{n}\right]-2(m_1x+m_2y)^2 \\ & \times\left[(m_1+n_1)x+(m_2+n_2)y+\frac{p}{n}\right]-4a(l_1x+l_2y)(m_1x+m_2y)=0 \end{aligned}$$

or

$$\begin{aligned} & (l_1x+l_2y)^2[(3m_1-n_1)x+(3m_2-n_2)y] \\ & +2(m_1x+m_2y)^2[(m_1+n_1)x+(m_2+n_2)y] \qquad A. \\ & +\frac{2p}{n}\left\{\left[(m_1x+m_2y)-\frac{na}{p}(l_1x+l_2y)\right]^2-\left[\frac{n^2a^2}{p^2}+\tfrac{1}{2}\right]\left[l_1x+l_2y\right]^2\right\}=0. \end{aligned}$$

If the new origin is taken at the point where the plane is cut by the perpendicular drawn to it from the old origin, the transformation scheme is

	x'	y'	$z'+p$
x	l_1	l_2	l
y	m_1	m_2	m
z	n_1	n_2	n

and the equation of the section is

$$\begin{aligned}(l_1x+l_2y+lp)^2[(3m_1-n_1)x+(3m_2-n_2)y+(3m-n)p]&\\ +2(m_1x+m_2y+mp)^2[(m_1+n_1)x+(m_2+n_2)y+(m+n)p]&\\ -4a(l_1x+l_2y+lp)(m_1x+m_2y+mp)=0,&\end{aligned} \qquad B.$$

the co-ordinates of the double point being

$$x=\frac{n_1}{n}p, \quad y=\frac{n_2}{n}p.$$

Equation B may be used in the cases in which A fails, *i.e.*, when the plane of section is parallel to the oblique axis, so that $n=0$.

If we choose the new x-axis to be perpendicular to the initial generator, we have $l_1=0$, $m_1m+n_1n=0$,

$$\therefore \quad \left.\begin{aligned} l_1 = 0\;,\; m_1 = \frac{n}{\lambda}\,,\; n_1 = -\frac{m}{\lambda}\,,\\ l_2 = -\lambda,\; m_2 = \frac{lm}{\lambda}\,,\; n_2 = \frac{nl}{\lambda}\,,\end{aligned}\right\}\text{where } \lambda=\sqrt{m^2+n^2}=\sqrt{1-l^2}\,.$$

Equation A then becomes

$$\begin{aligned}\lambda^4y^2[(3n+m)x+3(m-n)ly]+2(nx+lmy)^2[(n-m)x+(m+n)ly]&\\ +\frac{2p\lambda}{n}\left\{\left[nx+lmy+\frac{na}{p}\lambda^2y\right]^2-\left[\frac{n^2a^2}{p^2}+\tfrac{1}{2}\right]\lambda^4y^2\right\}=0.&\end{aligned} \qquad A_1$$

Also equation B becomes

$$\begin{aligned}\lambda^2(\lambda y-lp)^2[(3n+m)x+3(m-n)ly+(3m-n)\lambda p]&\\ +2(nx+lmy+m\lambda p)^2[(n-m)x+(m+n)ly+(m+n)\lambda p]&\\ +4a\lambda^2(\lambda y-lp)(nx+lmy+m\lambda p)=0,&\end{aligned} \qquad B_1$$

the co-ordinates of the double point being

$$x=-\frac{m}{n}\frac{p}{\lambda}, \quad y=l\frac{p}{\lambda}.$$

If we choose the new x-axis to be perpendicular to the oblique axis we have

$$l_1=\frac{m}{\nu}, \quad m_1=-\frac{l}{\nu}, \quad n_1=0,$$

$$l_2=\frac{nl}{\nu}, \quad m_2=\frac{mn}{\nu}, \quad n_2=-\nu.$$

Equation A then becomes

$$(mx+nly)^2[3lx-(3mn+\nu^2)y]+2(lx+mny)^2[lx+(\nu^2-mn)x] \qquad A_2$$
$$-\frac{2p\nu}{n}\left\{\left[(lx-mny)+\frac{na}{p}(mx+nly)\right]^2-\left[\frac{n^2a^2}{p^2}+\tfrac{1}{2}\right]\left[mx+nly\right]^2=0.\right.$$

Also equation B becomes

$$(mx+nly+l\nu p)^2[3lx-(\nu^2+3mn)y+(n-3m)p]$$
$$+2(lx-mny-m\nu p)^2[lx+(\nu^2-mn)y-(m+n)p] \qquad B_2$$
$$-4\nu a(mx+nly+l\nu p)(lx-mny-m\nu p)=0,$$

the co-ordinates of the double point being

$$x=0,\ y=-\frac{\nu}{n}p.$$

New Methods in the Geometry of a Plane Arc.*—I.

Cyclic and Sextactic Points.

By PROF. SYAMADAS MUKHOPADHYAYA, M.A.

The following paper introduces certain simple geometrical methods applicable to the general theory of plane curves. It brings into prominence, a certain class of singular points, on a plane curve, to which, it would appear, sufficient attention has not been given hitherto. If we suppose n consecutive points to travel steadily along a given curve, and carry on their shoulders, an osculating curve of a given kind, which varies continuously as it moves, then, upon the given curve, we shall usually have a number of places or singular points, where the osculating curve halts momentarily. At each halt, a change of shoulders is effected The rear-most moves off and the foremost receives an accession, or, the foremost goes away and the rear-most is strengthened. For the momen the osculating curve is borne on $n+1$ shoulders, that is, by one shoulder more than would suffice to carry its full weight.

A member of this class of singular points, I have, elsewhere, called *cyclic.* In the present paper, the cyclic and another member of the class the *sextactic,* have been studied together, more specially, in relation to a *simple oval,* that is, a closed convex curve, simply connected, on which the circle through any three points, as also, the conic through any five points, varies continuously. The conic, through any five points of the simple oval, we will suppose to be, an ellipse, *always,* for the purposes of this paper.

A *cyclic* point is a singular point, on a plane curve, where the circle of curvature passes through four consecutive points, instead of three. A *sextactic* point is a singular point, where the osculating conic passes through six consecutive points, instead of five. At a cyclic point, the circle of curvature may touch the given curve, internally or externally. In the former case, the point will be called *in*-cyclic and in the latter case, *ex*-cyclic. Similarly, at a sextactic point, the osculating conic may touch the given curve internally or externally. In the former case it may be allowable to call the point *in*-sextactic, and in the latter case *ex*-sextactic. With this much of introduction, we may proceed to demonstrate a number of interesting propositions.

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* For a first introduction to the new methods, the reader may be referred to a paper, by the present writer, in the Journal of the Asiatic Society of Bengal (New Series, Vol. IV., No. 8, 1908. Geometrical Theory of a Plane Non-Cyclic Arc, &c.).

Prop. I. If any circle meet a convex arc, at four points, O_1, O_2, O_3, O_4, then there must exist, a *cyclic* point on the arc, between the two extreme points O_1 and O_4, but not coinciding with O_1 or O_4.

Prop. II. If any conic meet a convex arc, at six points, O_1, O_2, O_3, O_4, O_5, O_6, then, there must exist a *sextactic* point on the arc, between the two extreme points O_1 and O_6, but not coinciding with O_1 or O_6.

In proposition I, we shall suppose that the circle through any three points of the arc varies continuously, as the points are moved, in any manner, along the arc. This, of course, implies that the radius of curvature varies continuously.

In proposition II, we shall suppose that the conic through any five points of the arc varies continuously as the points are moved, in any manner, along the arc. This conic must either be an ellipse, a parabola, or a hyperbola. In the last case, the five points of the arc must necessarily lie on the same branch of the hyperbola, for five points, distributed on two different branches of a hyperbola, cannot, evidently, lie on the same convex arc. For the purposes of this paper, we shall suppose that the conic through any five points of the arc, is always an ellipse, although this restriction is not *necessary* for proposition II.

To prove proposition I, it should be noticed that the four points O_1, O_2, O_3, O_4, determined by the intersection of a circle with the given arc, can be varied in position along the arc, by a continuous variation of the circle of intersection. Suppose we vary any two *adjacent* points O_1, O_2, by varying the circle, in such a way, that the remaining two points O_3, O_4, through which the circle passes, remain fixed. By this operation we can *draw together* O_1, O_2 as close as we like. When we thus draw together any two adjacent points O_1, O_2, it is to be understood, that they come indefinitely close, while O_3, O_4 remain fixed, but that they never overlap or cross each other or any other point, *e.g.*, O_3 or O_4. The *order* of the points O_1, O_2, O_3, O_4, is, therefore, strictly maintained. This will be obvious if we notice that circles through two fixed points cannot cross each other again.

Draw together, first, O_2, O_3 and then O_1, O_2 and then O_3, O_4, the remaining points, during each operation, continuing fixed. At the end of this cycle of three operations, O_1 and O_4 will have come closer together than at the beginning. By repeating this cycle a large number of times, we can bring the two extreme points 0_1, 0_4 as close together as we like, so that, ultimately, 0_1, 0_2, 0_3, 0_4 will have all come together at some point lying between the initial positions of 0_1 and 0_4. In fact, if 0_1, 0_2, 0_3, 0_4 do not come together ultimately, then there must be a minimum separation between 0_1 and 0_4. But this is impossible, for so long as the arc $0_1\,0_4$ is finite, it can be shortened by repeating the above-mentioned cycle of operations

The ultimate point, where 0_1, 0_2, 0_3, 0_4 all come together, will be *in-cyclic* or *ex-cyclic*, according as the circle $0_1\,0_2\,0_3\,0_4$ crosses in or crosses out at 0_4, *initially*. This will be so, because the order of the points 0_1, 0_2, 0_3, 0_4 is maintained during each operation. It is possible, however, that during an operation, an extra pair of intersections, say, X, Y, may arise between a pair of adjacent points, say, between 0_1 and 0_2. In that case we may drop 0_1 and X, and go on repeating our cycles on the shorter arc $Y\,0_2\,0_3\,0_4$. Evidently the circle $0_1\,X\,Y\,0_2\,0_3\,0_4$ will cross in or cross out at Y as it does at 0_1. If an extra intersection, say Z, exist *beyond* the extremities 0_1, 0_4, then during the cycles of operation it will always move *further* beyond.

The proof of proposition II is exactly similar, and similar observations apply to it. In this case, the cycle of operations may be described as follows: Draw together, first, 0_3, 0_4 and then, in succession, the pairs $(0_2, 0_3)$, $(0_4, 0_5)$, $(0_1, 0_2)$ and $(0_5, 0_6)$, the remaining four points, during each operation, continuing fixed. As conics through four points do not cross each other again, the order of the points 0_1, 0_2, 0_3, 0_4, 0_5, 0_6 will be strictly maintained during each operation.

Prop. III. On any simple oval, there must exist at least four cyclic points, two *in* and two *ex*.

Prop. IV. On any simple oval, there must exist at least six sextactic points, three *in* and three *ex*.

To prove Prop. III, draw a circle through any three points 0_1, 0_2, 0_3 on the oval. This circle must intersect the oval again, in a fourth point 0_4, as two closed figures intersect in an even number of points. Suppose the circle $0_1\,0_2\,0_3\,0_4$ crosses *in* and *out*, alternately, at 0_1, 0_2, 0_3, 0_4. To obtain the in-cyclic points, draw together 0_2, 0_3 at 0_2-0_3, and 0_4, 0_1 at 0_4-0_1, so that the circle 0_2-0_30_4-0_1 has internal double contact with the oval at 0_2-0_3 and 0_4-0_1. There must now exist an in-cyclic point, in each of the arcs 0_2-$0_30_4\,0_1$ and 0_4-0_10_2-0_3, by proposition I. Thus two in-cyclic points are established. Similarly, if we draw together 0_1, 0_2 at 0_1-0_2 and 0_3, 0_4 at 0_3-0_4, we shall have an ex-cyclic point, in each of the arcs 0_1-0_20_3-0_4 and 0_3-0_40_1-0_2.

Prop. IV. is proved in a similar way. Take any two equal parallel chords $0_1\,0_2$ and $0_3\,0_4$ in the oval. Then a conic through 0_1, 0_2, 0_3, 0_4

and any fifth point O_5 on the oval must be an ellipse, for two equal parallel chords cannot lie in the same branch of a hyperbola. Let the ellipse through O_1, O_2, O_3, O_4, O_5 meet the oval again at a sixth point O_6, for two closed figures must intersect at an even number of points. Suppose O_1, O_2, O_3, O_4, O_5, O_6 lie in order on the oval, and the conic through them crosses *in* and *out*, alternately, at O_1, O_2, O_3, O_4, O_5, O_6. To obtain the in-sextactic points, draw together O_1, O_2 at O_1-O_2, and then O_3, O_4, at O_3-O_4, and, finally, O_5, O_6, at O_5-O_6.

Then, the ellipse O_1-O_2 O_3-O_4 O_5-O_6 has internal triple contact with the oval at O_1-O_2, O_3-O_4, O_5-O_6. Therefore, from proposition II, we conclude, that there must be an in-sextactic point in each of the arcs O_1-O_2 O_3-O_4 O_5-O_6, O_3-O_4 O_5-O_6 O_1-O_2, O_5-O_6 O_1-O_2 O_3-O_4. Thus there will be, at least, two in-sextactic points on the oval. Let these two in-sextactic points be X, Y. Draw a narrow ellipse having internal double contact with the oval at X, Y. Let this ellipse grow, maintaining double contact with the oval at X, Y, till it touches the oval internally again at a third point Z, which may in special cases coincide with X or Y. Then, by proposition II, there must be another in-sextactic point in the arc $X\,Z\,Y$. Thus three insextactic points are demonstrated. In exactly similar way, three ex-sextactic points on the oval can be proved.

The following six propositions refer to arcs which are either non-cyclic or non-sextactic. A non-cyclic arc is one which does not possess a cyclic point in it, except it may be at the extremities. A non-sexactic arc is one which does not possess a sextactic point on it, except it may be at the extremities. On the non-cyclic arc we will suppose that the circle through any three points varies continuously. On the non-sextactic arc, we will suppose that the conic through any five points varies continuously and is, so far as this paper goes, always an ellipse.

Prop. V. If O_1, O_2, O_3 be any three points in order, on a non-cyclic arc, then the radius of the circle $O_1\,O_2\,O_3$ will continuously increase (or decrease), if the points O_1, O_2, O_3 be shifted in any manner, along the arc, in the same direction, provided the order of the points be maintained and the angle $O_1\,O_2\,O_3$ be never less than a right angle.

Prop. VI. If O_1, O_2, O_3, O_4, O_5 be any five points on a non-sextactic arc, then the area of the ellipse $O_1\,O_2\,O_3\,O_4\,O_5$ will continuously increase (or decrease), if the points be shifted, in any manner along the arc, in the same direction, provided the order of the points be maintained and the points be never so far separated from one another, that the elliptic arc $O_1\,O_2\,O_3\,O_4\,O_5$ exceeds the semi-ellipse.

To prove proposition V, suppose the points O_1, O_2, O_3 are shifted, one by one, in order, along the arc, in the same direction. Then during the shifting of each point, the radius will continually increase (or decrease).

If not, suppose, while O_2 is being shifted, O_1 and O_3 retaining their positions, the radius at first increases and then decreases, or at first decreases and then increases. Then O_2 will have two positions X, Y, between O_1, O_3, such that the radius of the circles O_1XO_3 and O_1YO_3 are equal. Therefore, we must have angles O_1XO_3, O_1YO_3 either equal or supplementary. But they cannot be supplementary, as then one of them will be acute, which is against hypothesis. Neither can the two angles be equal, for then the four points O_1, X, Y, O_3 would be concyclic and there would be a cyclic point on the given arc, which is also against hypothesis.

To prove proposition VI, suppose the points O_1, O_2, O_3, O_4, O_5 are shifted in order, one by one, in the same direction, along the arc. Then during each shifting, the area of the ellipse O_1, O_2, O_3, O_4, O_5 will continually increase or decrease If not, suppose, while any one point O_3 is being shifted, the others retaining their positions, the area at first increases and then decreases or at first decreases and then increases. Then O_3 will have two positions, X, Y, between O_2 and O_4, for which the area is the same, that is, the area of the ellipse O_1 O_2 X O_4 O_5 is equal to the area of the ellipse O_1 O_2 Y O_4 O_5. But it is easily shewn that the two areas cannot under the circumstances be equal (see following Lemma) unless the two ellipses coincide. Therefore O_1 O_2 X Y O_4 O_5 lie on the same ellipse, that is, there is a sextactic point on the given arc, which is against hypothesis.

Lemma. If O_1 O_2 X O_3 O_4 and O_1 O_2 Y O_3 O_4 be two elliptic arcs, each less than the corresponding semi-ellipse in length, then the area of the first ellipse will be greater than that of the second ellipse, provided arc O_2XO_3 pass above the arc O_2YO_3.

Project the system orthogonally, so that one of the ellipses becomes a circle, say, the second ellipse. Then, using the same letters for the projection, the circular arc O_1 O_2 O_3 O_4 is less than a semi-circle. The chords O_2 O_3 and O_1 O_4 will be either both perpendicular to the major axis, or both equally inclined to the major axis, of the projection of the first ellipse. In the former case, move O_2 O_3 parallel to itself till O_2 O_3 coincides with O_1 O_4. The circle O_1 O_2 O_3 O_4 continually increases in area during the operation, and ultimately has internal double contact with the ellipse. Hence, originally, the circle O_1 O_2 O_3 O_4 must have an area much less than that of the ellipse. If O_2 O_3 be not parallel to O_1 O_4, move O_2 O_3 parallel to itself towards O_1 O_4, till O_1 O_3 becomes parallel to O_2 O_4, and then it reduces to the former case. Of course, during this last operation, the area of the circle continually increases. Hence, in this case also, the circle has originally an area much less than that of the ellipse. But the ratio of areas is not altered by orthogonal projection. Hence the Lemma is proved.

Prop. VII. If O_1, O_2, O_3 be any three points, in order, on a non-cyclic arc, then the circle O_1 O_2 O_3 will always cross in at O_1 and O_3, or always cross out at O_1 and O_3, in whatever way we displace O_1, O_2, O_3, along the arc, maintaining their order.

Prop. VIII. If O_1, O_2, O_3, O_4, O_5 be any five points on a non-sextactic arc, then the conic $O_1 O_2 O_3 O_4 O_5$ will always cross in at O_1 and O_5, or always cross out at O_1 and O_5, in whatever way we displace the points O_1, O_2, O_3, O_4, O_5, along the arc, maintaining their relative order.

The above two propositions hardly need a formal proof. In proposition VII, the cutting in or cutting out at O_1 or O_3 can only be *screened*, if an extra-point of intersection X arise beyond O_1 or O_3. But this is impossible, as the arc is non-cyclic. Similar remarks apply to proposition VIII.

Prop. IX. If AB be a non-cyclic arc, in which any three points O_1, O_2, O_3 being taken, in order, the circle $O_1 O_2 O_3$ cuts in at O_1 and O_3, then the circle of curvature at A falls entirely within the circle of curvature at B.*

Prop. X. If AB be a non-sextactic arc, in which any five points O_1, O_2, O_3, O_4, O_5 being taken, in order, the ellipse $O_1 O_2 O_3 O_4 O_5$ cuts in at O_1 and O_5, then the osculating ellipse at A falls entirely within the osculating ellipse at B.

To prove proposition IX, move O_1, O_2, O_3, to A, so that we get the circle of curvature AAA at A, which falls below the arc AB. Similarly if we move O_3, O_2, O_1 to B we get the circle of curvature BBB at B which goes above the arc. Therefore, the circle AAA falls within the circle BBB, if we only consider portions above the chord AB. If we move O_3, O_2 to B and O_1 to A, we get the circle ABB, which falls below the arc and cuts AAA at some point O, above the chord AB. The circle ABB, which falls below the arc, touches at B the circle BBB, which goes above the arc. Therefore circle ABB falls within the circle BBB. Again the circle ABB cuts the circle AAA at A and O, therefore, below the chord AB, the circle AAA falls within the circle ABB, and, therefore, much more within the circle BBB. Thus the circle AAA falls within the circle BBB, both above and below the chord AB.

Analogous proof holds for proposition X. Bring O_1, O_2, O_3 to A, and O_4, O_5 to B. Then ellipse $AAABB$ falls below the given arc. If we bring down to A the other two points O_4, O_5, also, then the osculating ellipse $AAAAA$ will fall below the given arc and cut the ellipse $AAABB$ at some point O, above the chord AB but below the given arc. Therefore,

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[*] It has been noticed before by P. G. Tait, and comes easily by *assuming* the shape of the evolute between two centres of curvature O and C′, for if ρ and ρ' be the corresponding radii of curvature, then $\rho \sim \rho'$ is greater than the chord OC′ of the evolute. (Scientific Papers of P. G. Tait, Vol. II, page 403).

below the chord AB, the osculating ellipse $AAAAA$ falls within the ellipse $AAABB$, for these two ellipses have the four points A, A, A, C, common, and hence they cannot intersect again. Similarly, the ellipse $AABBB$ goes above the arc and cuts the osculating ellipse $BBBBB$, which also goes above the arc, at some point D, above the arc. Therefore, below the chord AB, the ellipse $AABBB$ falls within the ellipse $BBBBB$. But ellipses $AAABB$ and $AABBB$ have double contact at A and B, and the former goes below the arc and the latter above, therefore, the former $AAABB$ falls entirely within the latter $AAABB$. Hence below the chord AB, the ellipse $AAAAA$ falls within the ellipse $BBBBB$. Also, since the former goes below the arc, and the latter above, therefore above the chord AB, the ellipse $AAAAA$ falls within the ellipse $BBBBB$. Thus the osculating ellipse at A falls entirely within the osculating ellipse at B.

It may be pointed out that the director circle to the osculating ellipse at A falls entirely within the director circle to the osculating ellipse at B.

On a Non-Analytical Potential Function.

By GANESH PRASAD, M.A., D.Sc.

The object of this note is to discuss some of the properties of a non-analytical solution of

$$\frac{\delta^2 U}{\delta x^2}+\frac{\delta^2 U}{\delta y^2}=0,$$

first given by me in a paper[*] published in the *Mathematische Annalen*, Vol. 64, and to point out the importance of this solution with reference to the question of the nature of the solutions of partial differential equations of elliptic type.[†]

1. Representing by D a *closed* domain which is bounded by the straight lines

$$x=-1,\ x=1,$$
$$y=0,\ y=a>0,$$

the non-analytical potential function U may be defined for D in the following manner :—Let $f(x)$ be an even function which is equal to

$$\sum_{1}^{\infty}\frac{\phi(x-\omega_r)}{r^2}$$

for positive values of x, where $\{\omega_r\}$ is the aggregate of rational proper fractions, and $\phi(p)$ is

$$0 \text{ or } p^4 \cos\frac{1}{p}$$

according as p is 0 or different from 0 ; then

$$U=\frac{a_0}{2}+\sum_{1}^{\infty} a_m\, e^{-my} \cos mx,$$

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[*] "On a class of non-analytical surfaces of constant positive Gaussian curvature." (In German.)

[†] During the last 20 years the subject of the *analytical* nature of the solutions of partial differential equations of elliptic type has been investigated by a large number of writers, chiefly pupils of Professors Hilbert and Picard. The principal memoir on the subject is Bernstein's essay "Sur la nature analytique des équations aux derivées partielles du second ordre" (Math. Ann, Vol. 59, 1904). It is hardly necessary to mention that, in the present note, I consider the question of the nature of the solutions of such equations from a new point of view.

where

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x')\cos mx'\,dx'.$$

As shown in the paper cited above, U possesses the following properties in the domain D:—

(I) At all points

$$\frac{\delta U}{\delta x},\ \frac{\delta U}{\delta y},\ \frac{\delta^2 U}{\delta x^2},\ \frac{\delta^2 U}{\delta y^2}$$

exist; and the upper limits of

$$\left| U \right|,\ \left|\frac{\delta U}{\delta x}\right|,\ \left|\frac{\delta U}{\delta y}\right|,\ \left|\frac{\delta^2 U}{\delta x^2}\right|,\ \left|\frac{\delta^2 U}{\delta y^2}\right|$$

are less than a finite constant.

(II) At all points $U, \frac{\delta U}{\delta x}, \frac{\delta U}{\delta y}$ are continuous in (x, y).

(III) At all points, with the exception of the points of the aggregate

$$\{\omega_r, o\},\ \frac{\delta^2 U}{\delta x^2} \text{ and } \frac{\delta^2 U}{\delta y^2}$$

are continuous in (x, y); and, at every point (ω_r, o), $\frac{\delta^2 U}{\delta y^2}$ as well as $\frac{\delta^2 U}{\delta x^2}$ has a discontinuity, with respect to x, of the same kind as $\cos\frac{1}{p}$ has for $p=0$.

(IV) At all points

$$\frac{\delta^2 U}{\delta x^2}+\frac{\delta^2 U}{\delta y^2}=0.$$

Combining the properties (III) and (IV), we may make the following statement:—*Although there are an infinite number of points of discontinuity of* $\frac{\delta^2 U}{\delta x^2}$ *and* $\frac{\delta^2 U}{\delta y^2}$ *in the domain D, at all points of this domain (the points of discontinuity being included),*

$$\frac{\delta^2 U}{\delta x^2}+\frac{\delta^2 U}{\delta y^2}=0.$$

2. I proceed now to consider the bearing of the above statement on the following question*:—*Is it possible to obtain such a solution u of the equation of elliptic type*

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* Cf. Prof. Baire's discussion of a similar question, relating to the equation

$$\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}=0,$$

n his memoir "Sur les fonctions de variables réelles" (Annali di Matematica, 1899).

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=F\left(x,\ y,\ u,\frac{\delta u}{\delta x},\ \frac{\delta u}{\delta y}\right)\ldots\ldots (1)$$

that the domain, at every point of which this equation is satisfied, contains, nevertheless, an infinite number of points at which $\frac{\delta^2 u}{\delta x^2}$ and $\frac{\delta^2 u}{\delta y^2}$ *are discontinuous?*

The answer to the above question must be in the affirmative. In order to see this, it should be first noted that, according to Picard's well-known method of successive approximations,

$$u = U + \sum_{2}^{\infty} \nu_n$$

is a solution of (1), where

$$\nu_2 = -\frac{1}{2\pi}\iint F\{x',\ y',\ U(x',\ y'),\ldots\}\,G(x,\ y,\ x',\ y')\,dx'\ dy'$$

and, for $n > 2$,

$$\nu_n = -\frac{1}{2\pi}\iint\left[F\left\{x', y', u_{n-1}\ (x', y'),\ldots\right\}-F\left\{x', y', u_{n-2}(x', y'),\ldots\right\}\right]$$
$$\times\, G(x,\ y,\ x',\ y')\ \ dx'\ dy',$$

u_n standing for

$$U + \sum_{2}^{n}\ \nu_r\,,$$

$G(x,\ y,\ x',\ y')$ for the Green's function for the domain D, and the integrations being extended over the domain D. Now, as proved in § 3 of my paper, cited above, in virtue of the continuity of U, $\frac{\delta U}{\delta x}$ and $\frac{\delta U}{\delta y}$, the second differential co-efficients of the ν's are expressible in terms of the first differential co-efficients of U and are, consequently, continuous at every point of D. Therefore it follows that $\frac{\delta^2 u}{\delta x^2}$ and $\frac{\delta^2 u}{\delta y^2}$ are discontinuous exactly in the same manner as $\frac{\delta^2 U}{\delta x^2}$ and $\frac{\delta^2 U}{\delta y^2}$. Thus, for the domain D, a solution u of the equation (1) has been obtained such that, at every point $(\omega_r,\ 0)$ of this domain, $\frac{\delta^2 u}{\delta x^2}$ as well as $\frac{\delta^2 u}{\delta y^2}$ is discontinuous with respect to x.

Summary of Principal Mathematical Journals.

Mathematische Annalen, vol. 66, part 1 (Aug., 1908).

Emil Hilb of Erlangen.—On representations of arbitrary functions by means of integrals. (In German.)

During the last five years, the question of the expansibility of arbitrary functions has been discussed by Hilbert and his pupils; their treatment being based on the investigation of the integral equation

$$f(s)=\phi(s)-\lambda\int_a^b K(s,\,t)\ \phi(t)dt,$$

where the kernel $K\ (s,\,t)$ is symmetrical in s and t and satisfies the condition that

$$\int_a^b\int_a^b \{K\ (s,\,t)\}^2\,ds\,dt$$

is finite. The object of the present memoir is to discuss the question of the representations of arbitrary functions by means of the characteristic functions of such differential equations as possess singular points; the corresponding kernel being no longer such that

$$\int_a^b\int_a^b \{K\ (s,\,t)\}^2\,ds\,dt$$

is finite. The author bases his discussion on Hilbert's memoirs on the theory of quadratic forms of an infinite number of variables, and succeeds in obtaining the known, as well as some new, generalizations of Fourier's Integral Theorem. (p. 1—66).

Fritz Jerosch (deceased) and *Hermann Weyl* of Göttingen.—On the convergence of series of periodic functions. (In German.)

The authors first prove the following theorem: A series $\sum_1^\infty c_n \cos nx$ converges for all values of x with the exception of such values as belong to a certain aggregate of measure 0; provided that it is possible to find a positive number C and an exponent $\gamma > \frac{2}{3}$, so that, for every value of n,

$$\left| c_n \right| \leqq \frac{C}{n^\gamma}.$$

They then give a generalization of this theorem for the series

$$\sum_1^\infty c_n \, \phi(nx),$$

where $\phi(y)$ is expansible in the form

$$\sum_1^\infty (a_n \cos ny + b_n \sin ny). \quad \text{(p. 67—80).}$$

G. Faber of Carlsruhe.—On continuous functions. (In German.)

The object of this paper is to point out that it is possible, by means of the superposition of broken straight lines, to construct examples of continuous functions which possess particular properties, *e.g.*, functions whose non-differentiability, up to a certain degree, can be represented geometrically. (p. 81—94).

Arthur Wieferich of Munster.—Proof of the theorem that any integer is representable as the sum of not more than nine positive cubes. (In German.) (p. 95—101).

Edmund Landau of Berlin.—On an application of the theory of prime numbers to Waring's problem in the elementary theory of numbers. (In German.)

Combining Wieferich's method with a new theorem, first given by de la Vallée Poussin in 1896, the author proves that, if an integer exceeds a certain integer, it is representable as the sum of not more than *eight* positive cubes. (p. 102—105).

Arthur Wieferich of Munster.—On the representation of numbers as sums of fourth powers. (In German.)

It is proved that any integer is representable as the sum of 37 fourth powers. (p. 106—108).

J. Ouspensky of St. Petersburg.—Note on the integral numbers which depend on a fifth root of unity. (In French.)

The author gives a simple demonstration of Euclid's algorithm for such numbers. (p. 109—112).

W. Fr. Meyer of Königsberg.—On an application of the theory of invariants to the expansion of integrals—specially rational, elliptic and hyperelliptic integrals—in series. (In German.)

The author found that, in virtue of the fact (first pointed out by Prof. Klein) that the elliptic integral

$$J(x) = \int \frac{dx}{\sqrt{f_4(x)}} = \int \frac{dx}{\sqrt{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}}$$

is a transcendental covariant of the binary biquadratic form $f_4(x)$, the coefficients in the expansion of $J(x)$, in ascending powers of x, obey a very simple law. He then successfully attempted to extend this law to higher integrals of the form

$$\int dx\, f_n^{\mu}(x)\, g_p^{\nu}(x)\, h_q^{\pi}(x),\ldots,$$

where $f_n(x)$, $g_p(x)$, $h_q(x)$, ... represent a finite number of binary forms of orders $n, p, q, \ldots$, and $u, \nu, \pi, \ldots$ are arbitrary exponents (real or complex)

The present memoir contains the results of the author's investigation. He first considers the general case, and then deduces the expansion of $J(x)$ and some other interesting results including the following known expansion:—

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \sum_{l=0}^{\infty} \sum_{s=0}^{l} x^{2l+1}\, k^{2s}\, \frac{1}{2l+1}$$

$$\frac{1\cdot3\cdot5\ldots\{2(l-s)-1\}}{2\cdot4\cdot6\ldots\{2(l-s)\}} \times \frac{1\cdot3\cdot5\ldots(2s-1)}{2\cdot4\cdot6\ldots2s}.$$

(p. 113–132).

G. A. Miller, of Urbana, U.S.A.—On the multiple holomorphs of a group. (In English.)

The object of the present paper is to examine the abelian groups with respect to multiple holomorphs. (133—142).

Prize Proclamation of the Royal Society of Sciences of Göttingen for the proof of Fermat's theorem. (In German.) (p. 143—144).

Mathematische Annalen, vol. 66, parts 2 and 3 (Nov. and Dec., 1908).

Ernst Jacobsthal, of Berlin.—On the construction of transfinite Arithmetic. (In German.)

With the help of the theorem of transfinite induction, the author defines certain operations which include, as particular cases, addition, multiplication and involution The course of these operations is described by putting down certain inequalities ; and it is shown that, corresponding to each operation, there are certain numbers—called the *Hauptzahlen* or chief numbers of the operation—which are of special importance. The results relating to these Hauptzahlen are used to discuss the notions of prime numbers and of the divisors and multiples of given numbers. It is found that addition is the most important of all the operations, just as in finite Arithmetic the operation of multiplication is the most important. (p. 145—194).

Georg Landsberg, of Kiel.—On the class of those surfaces which cut a bundle of rays at a constant angle. (In German.)

Denoting by r, θ, ϕ the polar co-ordinates of a point, and taking

$$r = \omega\,(\theta, \phi)$$

to be the equation of such a surface, the author finds the partial differential equation of the class to be

$$\left(\frac{\delta \log r}{\delta\,\theta}\right)^2 + \left(\frac{\delta \log r}{\sin\theta\delta\,\phi}\right)^2 = \cot^2 h,$$

where h is the constant angle. He then deduces certain peculiarities of the surfaces, and the different possible ways of generating them. The chief result is this: Every surface of this class can be generated by the motion of a logarithmic spiral, whose plane rolls, without sliding, along a cone having its vertex at the pole of the spiral, but, otherwise, quite arbitrary. (p. 195—201).

Guido Fubini, of Genoa.—Applications of the theory of continuous groups to differential geometry and to the equations of Lagrange. (In Italian.)

After a paragraph devoted to preliminary formulæ, the author investigates, in the next four paragraphs, the various metrical geometries for which there exists a continuous group of movements, a continuous conformal group, a continuous geodesic group, or a continuous group whose transformations change any hyper-sphere into another hyper-sphere. The last paragraph is devoted to the applications of the preceding results to the equations of Lagrange. (p. 202—214).

E. Hilb, of Erlangen.—On Klein's theorems in the theory of linear differential equations. (In German.)

In his lectures on linear differential equations of the second order, delivered in the years 1890—91 and 1893—94, Prof. Klein discusses the following question: Given the real singular points and the corresponding indices, how far is it possible to determine the unknown parameter so that the polygon, bounded by circular-arcs, on which the half-plane, on one side of the axis of real quantities, is represented by means of the quotient of two particular solutions of the differential equation, may have prescribed properties? The present paper, which was written under Klein's inspiration, has for its object the proving of certain assertions, about this question, made by Klein in his paper in the Math. Ann., vol. 64. The author bases his proofs on certain elementary considerations of continuity which are quite analogous to those occurring in the theory of the real roots of algebraical equations. (p. 215—257).

Charles Haseman, of Bloomington, U.S.A.—Application of the theory of integral equations to some boundary value problems in the theory of functions. (In German.)

In § 1 the following problem is solved: For the inside of a given region, determine a regular analytical complex function,

$$f(z) = u(x, y) + iv(x, y),$$

of which the real and imaginary parts, $u(s)$ and $v(s)$, on the closed boundary-curve C, satisfy the condition

$$a(s)\,u(s) + b(s)\,v(s) + c(s) = 0,$$

where $a(s)$, $b(s)$, $c(s)$ are given real functions of the arc s of C and, although subject to certain conditions, are not necessarily continuous at every point of C. In § 2 the author finds two functions, for the inside of a given region, when two relations between their real and imaginary parts on the boundary are prescribed; and in § 3 he considers a similar problem for the case when one function is required for the inside, and the other for the outside, of a given region. (p. 258—272).

H. Weyl, of Göttingen.—Singular integral equations. (In German.)

The method of an infinite number of variables was first successfully used by Hilbert in the theory of the integral equation

$$f(s) = \varphi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

where $f(s)$ and the kernel $K(s, t)$ are given functions, and $K(s, t)$ is continuous. In the first part, it is shown that this method is also applicable to cases in which $K(s, t)$ is not necessarily continuous at every point (s, t); and in the remaining part, by using special discontinuous kernels, the author deduces (1) some interesting results about the expansibility of an arbitrary function in series of Hermite's orthogonal functions and in series of Laguerre's polynomials, and (2) some integral theorems including the following: When $f(s)$ satisfies certain conditions of continuity and differentiability,

$$f(s) = \tfrac{1}{2}\int_1^\infty \sqrt{s}\, J_\nu\left(s\sqrt{\lambda - 1}\right) \int_0^\infty f(t) \sqrt{t}\, J_\nu\left(t\sqrt{\lambda - 1}\right) dt\, d\lambda. \quad \text{(p. 273—324).}$$

Wera Myller-Lebedeff, of Bucharest.—On the application of integral equations to a parabolic boundary-value problem. (In German.)

The author investigates a continuous solution of the equation of parabolic type

$$\frac{\delta^2 u}{\delta x^2} + \frac{1}{x}\frac{\delta u}{\delta x} - \frac{1}{x}\frac{\delta u}{dt} = 0$$

for a domain $A\,B\,C\,D$, bounded by

$$x = \xi_1(t),$$
$$t = a,$$
$$x = \xi_2(t),$$
$$t = b;$$

where the values of u on $A\,B$, $B\,C$ and $C\,D$ are prescribed to be $f_1(t)$, $f(x)$ and $f_2(t)$ respectively, f_1, f and f_2 being continuous and differentiable. (p. 325—330).

E. Study, of Bonn.—On the real solutions of the equation

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0. \quad \text{(In German.)}$$

In this paper, which is an extension of the author's note in the Math. Ann., Vol. 63, the following theorem is proved : The real part $u\ (x, y)$ of an analytical function

$$w = f(x+iy)$$

has then, and only then, a connection with another real function u^* of the real variables x and y, when the inverse function of w has a real period ; further, if $2\ \omega$ is a positive primitive period of the inverse function, then corresponding to this period there exists a function u^*, which stands to the given function u in the relation that each determines the other up to a multiple of $2\ \omega$. The author also proves that, if u be a real solution of

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0,$$

then the locus of the point whose cartesian co-ordinates are (x, y, u) is a surface of translation. (p. 331—336).

Walter Schnee, of Berlin.—About the problem of the convergence of Dirichlet's series. (In German.)

As a generalization of a theorem of Landau's, the author proves the following theorem : Let

$$f(s)=\sum_{n=1}^{\infty} b_n e^{-\lambda_n s},$$

where

$$s=\sigma+it,$$

and $\{\lambda_n\}$ denotes an arbitrary sequence of positive numbers, monotonously increasing up to ∞ in such a manner that (1) the superior limit of $\frac{\log n}{\lambda^n}$ is a finite quantity a and (2) corresponding to every number $\delta>0$ a whole number ρ_0 exists such that

$$\lambda_{\rho+1}-\lambda_\rho > e^{-(a+\delta)\,\lambda_\rho}$$

when $\rho \geqq \rho_0$. Further, corresponding to every number $\delta > 0$, let there exist a whole number n_0 such that

$$\left|b_n\right| < e^{\delta\lambda_n}$$

when $n \geqq n_0$. Also let the function represented by the series $f(s)$ be regular for $R(s) > \eta$, where η is an arbitrary real number less than a; and, finally, for

$$\sigma > \eta, |t| \geqq 1,$$

let

$$|f(s)| < B|t|^k,$$

where B is a constant and

$$0 \leqq k < 1.$$

Then the series $f(s)$ converges at least for

$$R(s) > \frac{\eta + ka}{1+k}. \quad \text{(p. 337—349).}$$

G. Hamel, of Brünn.—On the foundations of Mechanics. (In German.)

This paper is an attempt at a rigorous foundation of classical Mechanics. Although the author recognizes the fundamental importance of the works of Kirchoff, Mach and Poincaré, he does not accept, in their entirety, the opinions of these writers. (p. 350—397).

Francesco Aurelio Dall' Acqua of Mantua.—On the integration of the Hamilton-Jacobi equations by the method of separation of variables. (In Italian.)

In the Math. Ann., Vol. 42, Stäckel proposed the following problem: Under what conditions are the Hamilton-Jacobi equations, which are given by

$$H\,(p_1, p_2, \dots, p_n\,;\, x_1, x_2, \dots, x_n) = \text{const.}$$

$$\left(p_h = \frac{\delta W}{\delta x_h},\ \ h = 1, 2, \dots n\right),$$

integrable by the method of separation of variables, when H corresponds to a dynamical problem with connections independent of time? The author obtains, in a direct manner, the conditions and the complete solution for the case of three variables. (p. 398—415).

Journal de Mathématiques.—(Liouville's Journal), Paris, 1908, sixth series, fourth volume, Nos. 1 and 2.

L. Remy.—On certain algebraic surfaces connected with the Abelian functions of class 3.

Mons. Remy puts a general problem of a particular type concerning algebraic surfaces in the introduction to his paper, *viz.*, how a correspondence may be established between algebraic surfaces and curves of class p such that couples of points on the latter may be represented by points on the former. He, however, restricts himself to the case of $p=3$ and studies the surfaces such as S_1 to a point on which correspond two couples of points on a curve C_1 situated in a straight line. The general theorems concerning such surfaces are applied to a particular surface S_1 of the sixth degree. The paper ends with the study of the class of curves traced on S_1. The surface S_1 is an envelope of a certain family of cubic surfaces, and its equation may be put in the form $F[(Pi\ Qa,i), (Pj\ Qa,j), (Pk\ Qa,k)]=0$ where $Pi=0$ are planes and $Qa.i=0$ are quadrics, which together form a degenerated cubic surface circumscribed to S_1.

S_1 has three concurrent double lines and five singular tangent planes touching it along a plane cubic.

A. Buhl.—On the generalisation of trigonometric series. The author shows how to form series more general than the trigonometric series, which contain an arbitrary parameter ψ and which represent $f(x)$ in the first interval; then this function multiplied by $\cos\psi$, $\cos 2\psi, \ldots$, or $\sin\psi$, $\sin 2\psi, \ldots$ in the intervals following the first and of identical length, and so on. In fact he studies this general question: Given a trigonometric series representing a well-defined function, is it possible to find a simple and immediate operation that will transform this series into an infinity of others, which will still define the same function in the same intervals? The answer is in the affirmative; in particular, he shows how a series can be obtained which, though not a Fourier series, will behave exactly like it, and will represent a given function for a given interval. He applies the result to solutions of partial linear differential equations, in particular, to Laplace's equation in two dimensions.

C. Popovici.—On the equations with reciprocal integrals.

Two systems of differential equations

$$(1)\quad \frac{dx_1}{u_1}=\ldots\ldots=\frac{dx_{n-1}}{u_{n-1}}=dx_n$$

$$(2)\quad \frac{dx_1}{v_1}=\ldots\ldots=\frac{dx_{n-1}}{v_{n-1}}=dx_n$$

are called equations with reciprocal integrals, where $v_1\ldots\ldots v_{n-1}$ are the fundamental system of first integrals of equations (1), while $u_1\ldots\ldots u_{n-1}$ are those of (2). The results are applied to the theory of groups and to problems of mechanics. One result is that a system of equations

$$\frac{d^2x_i}{dt^2}=X_i(x_1,\ldots\ldots x_n, t),\ i=1,\ldots\ldots n$$

do not admit of a system of immediate integrals of the form

$$X_k\,(x_i, \ldots\ldots x_n, t) = L_k\left(\frac{dx_1}{dt}, \ldots\ldots \frac{dx_n}{dt}\right),$$

depending on arbitrary functions and constants.

Haton de la Goupillière.—Note on the principal axes of the times of *parcours*.

T. Lalesco.—On the equation of Volterra.

The aim of this paper is to develop and complete in certain points the beautiful researches of Volterra in the functional equation

$$\int_a^x f(x, s)\,\phi(s)\,ds = F(x),$$

and in analogous equations where one at least of the limits of the integral is variable; $\phi(s)$ denotes the unknown function. A result worthy of notice is the derivation of the well-known expression

$$\int_0^\infty u^{x-1}\,e^{-u}\,du \text{ for } \Gamma(x)$$

from the simple type of

$$f(x, s),\ viz.,\ f(0, 0) = 0$$

occurring in the general functional equation of Volterra.

Annals of Mathematics.—Published under the auspices of Harvard University, second series, vol. 9, Nos. 2, 3.

R. D. Carmichael.—On the classification of plane algebraic curves possessing four-fold symmetry with respect to a point. C. E. Stromquist.—A second inverse problem of the Calculus of Variations. H. C. Wolff.—The continuous plane motion of a liquid bounded by two right lines. J. K. Whittemore.—A problem in Chance. A. E. Kennelly.—The expression of constant and of alternating continued fractions in hyperbolic functions. O. D. Kellog.—A necessary condition that all the roots of an algebraic equation be real. E. B. Wilson.—The equilibrium of a heavy homogeneous chain in a uniformly rotating plane. J. L. Coolidge.—The continuity of the roots of an algebraic equation. W. F. Osgood.—On the differentiation of definite integrals. H. E. Buchanan and J. H. Hildebrandt.—Note on the convergence of a sequence of functions of a certain type. G. A. Bliss.—On the inverse problem of the Calculus of Variations. C. N. Haskins.—A geometrical interpretation of generalised law of the mean. P. Saurel.—On the torsion of a curve.

Societies and Academies.

Academy of Sciences, Paris, 1908.

January—June.

A. Buhl.—On the summability of Fourier series.

A. Denjoy.—On the choice of the index of convergence for infinite integral series.

Schlesinger.—On a differential system of the second degree.

Esclangon.—On the periodic solutions of certain functional equations.

Eug. E. Levi.—On the equation

$$\delta^2 z \mid \delta x^2 - \delta z \mid \delta y = 0.$$

E. Cartan.—On the definition of the area of a portion of a curved surface.

Tzitzéica.—On a class of surfaces.

Léop. Téjer.—On the development of an arbitrary function in terms of Laplacians.

Petrovitsch.—A theorem on Taylor's series.

E. Cotton.—On the approximate integration of differential equations.

E. Goursat.—On a theorem in the theory of integral equations.

E. Holmgren.—Remark on a communication of Levi.

Remoundos.—On the singularities of differential equations of the first order.

Popovici.—On the congruences of plane curves.

Raffy.—On the surfaces with non-distinct lines of curvature.

Störmer.—Case of reduction of differential equations of the trajectory of an electric corpuscle in a magnetic field.

Traynard.—On a hyperelliptic surface of third order.

Kolosoff.—On the problem of elasticity in two dimensions.

Nordmann.—New researches on variable stars.

A. Buhl.—On the series of Taylorian polynomes.

Korn.—General solution of the problem of equilibrium in the theory of elasticity when the forces acting on the surface are given.

Zaremba.—On the application of a problem alternative to the bi-harmonic problem.

Raffy.—On the surfaces having coincident lines of curvature.

C. Störmer.—On the differential equations of an electrified corpuscle in a magnetic field.

Le Vavasseur.—On the sub-groups of the linear homogeneous group of four variables and the corresponding systems of partial differential equations.

Raffy.—On the *persistant* conjugate réseaux which includes a family of minimum lines.

G. Darboux.—On a theorem relating to the theory of twisted curves.

Auric.—On entropy.

Humbert.—Formulas relating to the minima of classes of quadratic and positive binary forms.

Krygowski.—On the canonical hyperbolical integrals of the second type.

P. Duhem.—On the discovery of the law of falling bodies.

P. Zervos.—On a method of Goursat in the solution of the Problem of Monge.

L. Bachelier.—The general problem of probabilities of repeated trials.

Jacob.—Integrometer for integrating Abel's equation.

Auric.—On the devélopment of an algebraic number in continued fractions.

E. Picard.—On a partial differential equation relating to a closed surface.

De Séguier.—On the bilinear forms.

Sanielovici.—On the partial differential equations of vibrating membranes.

E. Borel.—On the analysis of polymorphic curves.

A. Demoulin.—On the ruled surfaces.

A. Denjoy.—On the canonical products of the infinite kind.

Sanielivici.—On the partial differential equations of vibrating membranes.

Royal Society, London, 1909.

January—March.

A. E. H. Love.—The yielding of the earth to disturbing forces.

J. Larmor.—The Relation of the Earth's free precessional nutation to its resistance against tidal deformation.

A. H. Gibson.—On the depression of the filament of maximum velocity in a stream flowing through an open channel.

N. Bohr.—Determination of the surface tension of water by the method of jet vibration.

G. H. Darwin.—Note on the stability of Jacobi's ellipsoid.

London Mathematical Society, 1909.

January—March.

H. Hilton.—The canonical form of a linear substitution.

J. Hammond.—On the solution of the quintic.

Lt.-Col. A. Cunningham.—On octavic and sexdecimic residuacity.

E. W. Hobson.—On change of the variable in a Lebesgue integral.

F. H. Jackson.—On Abel's extension of Taylor's series.

H. M. Macdonald.—Note on the evaluation of a certain integral containing Bessel's Functions.

A. C. Dixon.—On the relations between Pfaff's Problem and the Calculus of Variations.

W. H. Young.—On Implicit functions and their differentials.

W. H. Salmon.—On a certain family of cubic surfaces.

E. W. Hobson.—Some fundamental properties of Lebesgue integrals in two dimensional domain.

L. E. Dickson.—Modular invariants of a general system of linear forms.

H. Bateman and *E. Cunningham.*—The conformal transformations of a space of four dimensions and the generalisation of the Lorentz Einslein principle.

W. H. Young.—On indeterminate forms.

J. Larmor.—The kinetic image of a convected electric system in a conducting plane sheet.

G. H. Hardy.—On an integral equation.

H. Bateman.—The transformation of the electro-dynamical equations and the laws of motion.

E. Cunningham.—The transformation of the electro-dynamic equations of moving bodies.

American Mathematical Society.

April 1907 to April 1908.

F. L. Griffin.—On the Apsidal angle in central orbits.

G. A. Miller.—The Invariant substitutions under a substitution group.

V. Snyder.—On a special algebraic curve having a net of minimum adjoint curves.

R. D. Carmichael.—Note in certain inverse problems in the simplex theory of numbers.

G. A. Miller.—Third report on recent progress in the theory of Groups of Finite Order.

L. E. Dickson.—On quadratic forms in a general field.

D. C. Gillespie.—On the canonical substitution in the Hamilton-Jacobi Canonical system of differential equations.

L. E. Dickson.—On triple algebras and ternary cubic forms.

E. Kasner.—Isothermal systems in Dynamics.

C. H. Sisam.—On the equations of quartic surfaces in terms of quadratic forms.

A. S. Chessin.—On an integral appearing in Photometry.

J. I. Hutchinson.—Hermitian forms with zero determinant.

C. A. Noble.—Singular points of a simple kind of differential equation of the second order.

G. W. Hill.—Subjective Geometry.

L. E. Dickson.—On higher congruences and modular invariants.

Max Mason.—Note on Jacobi's equation in the Calculus of Variations.

E. R. Hedrick.—On the distance from a point to a surface.

L. I. Neikirk.—Geometric representation of the Galois field.

W. B. Fite.—Concerning the degree of an irreducible linear homogeneous group.

F. R. Sharpe.—On the Lorentzian transformation and the radiation from a moving electron.

J. W. Young.—A fundamental invariant of the discontinuous ζ-groups defined by the normal curves of order n in a space of n dimensions.

C. N. Moore.—On certain constants analogous to Fourier's constants.

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Reviews.

"A Course of Pure Mathematics."—By Professor G. H. Hardy. Cambridge University Press, 1908, pp. i to x, 428. Price 12 shillings, net.

This book is divided into ten chapters. "Mathematical analysis being essentially the Science of the Continuum, it is proper that every course in analysis should begin logically with the study of irrational numbers." In the first chapter are accordingly discussed the various classes of numbers included in the Arithmetical continuum. Although no attempt has been made to give a complete account of a purely arithmetical theory of irrational number, for which the reader is to turn to the fascinating pages of Dedekind's Essays on Number (translated by Beman Smith: Open Court Publishing Coy.) and of Dr. Young's treatise on "Sets of Points" (Cambridge University Press), sufficient information has been given in the eight sections to whet the reader's appetite for the magnificent theories of Cantor and Dedekind. The linear continuum is taken for granted, and the existence of a definite number corresponding to each of its points is assumed and the various classes of numbers are analysed, mostly from the commonsense point of view. The chapter ends with a definition of the "continuous real variable."

Chapter II deals with "functions of real variables." The idea of functional dependence has been fully illustrated, and a systematic and detailed account has been given of most of the important classes of functions.

Chapter III contains an account of what is usually styled "complex number." A good deal of important and valuable matter regarding Vector analysis, bilinear transformation, coaxial circles, and cross-ratios, has been given in place of purely formal consequences of DeMoivre's Theorem. Professor Hardy's criticism in section 34 that a complex-number, $x+yi$, (the usual synonym, "imaginary-number," is by no means a happily chosen one) is really not a number at all but a *pair of numbers* (x, y) united symbolically, for purposes purely of convenience, and his statement in the same section that "the field of complex numbers is closed for algebraical operations" are noteworthy.

Chapter IV deals with "limits of functions of a positive integral variable." This chapter is one of the principal features of the book. The notion of a limit is a difficult one and its importance can never be exaggerated. Professor Hardy has done a real service to students aspiring to be mathematicians, by taking pains to make this chapter as full and interesting as possible. Before giving a formal definition of a limit, the author has descanted with admirable clearness on the use of such terms or phrases as "large," "infinity," "n tends to infinity," &c., and the author's treatment of them recalls to mind the masterly exposition of the same subjects, two generations ago, in the pages of Augustus DeMorgan's Calculus (Introductory Chapter). Two definitions are given, *viz.*: (1) of a finite limit, and (2) of an infinite limit (positive infinity), with geometrical illustrations; and some important points are considered concerning these definitions, among which the fact that "a limit is not the value of a function," can hardly be too strongly impressed on a student of Mathematics. "Oscillating" functions are then defined as well as finite and infinite oscillations, with illustrations. The behaviour of the sum, product, difference and the quotient of two functions whose behaviour is known as regards limits, are treated with logical rigour and the warning finger is pointed at obvious pitfalls; for instance, a rigorous proof in some detail is given of the well-known theorem "if $\phi(n)$ and $\psi(n)$ tend to limits a, b, then $\phi(n)+\psi(n)$ tends to the limit $a+b$," and the rigorous proof in place of a commonsense one is supported by the remarks:—"In this case the result obviously indicated by *commonsense* was true. In more difficult cases commonsense as often indicates an untrue result as a true one; sometimes it fails to give any indications at all. In such cases vague general arguments are worse than useless; they lead to mistakes not only gross in themselves but entirely confusing in their consequences. And unless the reader is prepared to take the trouble to try and understand the way in which rigorous methods apply to simple and obvious cases, when their application is easy, he will find that when he comes to difficult questions which cannot be settled without them, he has not the capacity to use them." No remarks could be more just than these, and it would not be paradoxical to say that the student of the present day mathematics "must learn to think in the ϵ, δ forms of Cauchy and Weierstrass." The special but particularly important class of functions of n whose variation, as n tends to Infinity, is always in the same direction, is then dealt with by means of a definition and a theorem with corollaries, the great importance of which lies in the fact that they give us a means of deciding, in great many cases, whether a given function of n does or does not tend to a limit as n tends to Infinity, *without requiring us to be able to guess or otherwise infer beforehand what the limit is.*

The consideration of limits naturally leads to the theory of infinite series and some general theorems concerning them. The theory of decimals in general, recurring and non-recurring, and the expression of irrational numbers as non-recurring decimals, are given in smaller type as examples; for a full exposition of the subjects, the reader must turn to the informing Appendix I to Professor Bromwich's Introduction to the Theory of Infinite Series (1908, Macmillan & Co.) which gives the Arithmetic theory of irrational number and limits.

Chapter V is no less important than Chapter IV. In it are discussed limits of functions of a continuous variable, continuous and discontinuous functions. This chapter is supplementary to Chapter IV, and the close correspondence between functions of a *continuous* real variable and functions of a positive integral variable, is indicated, and we are shown how the definitions and theorems to which we are led in Chapter IV, may be repeated for purposes of Chapter V, by reading x for n with this difference, that x varies through all values as it tends to Infinity, whereas n tends to Infinity by a series of jumps. Section 83 of this chapter is specially valuable in that it shows where a beginner (and, for the matter of that, an advanced student also) is likely to commit an error. Thus in Limit $x \longrightarrow 0\ (a+bx+cx^2+\ldots kx^m)$ one is likely to put $x=0$ at once. Surely the correct *result* is obtained but the procedure is wrong; for the statement

$$\text{Lim}_{x \longrightarrow 0}\phi(x) = l$$

is a statement about the values of $\phi(x)$ when x has any value distinct from but *differing by little from zero.* It is *not* a statement about the *value* of $\phi(x)$ when $x=0$. To emphasise this important distinction, the values of x are, in the definition of a limit $(x \longrightarrow 0)$, defined by the inequalities $0 < x \leqq \eta$. Thus take the function x/x. This is not defined at all for $x=0$ while the limit is $=1$, so long as x differs from zero, *however small the difference might be.*

The rigorous definition of limits leads to the definition of a continuous function of a real variable (the continuity may be through an interval or everywhere), and the definition is illustrated geometrically. From continuity the author proceeds to the classification of discontinuities. In section 86, Professor Hardy puts what is the commonsense notion of continuity and reconciles it with the mathematical definition, thereby showing that our commonsense notion is substantially accurate and capable of precise statement in mathematical terms. The chapter closes with an account of inverse functions and of the range of values of a continuous function, the theorems regarding the latter being of an abstract character.

A careful study of chapters IV and V prepares the student for admission to the portals of the calculus, *viz.*, derivatives and integrals, the subject matter of chapter VI. The main standpoints of the differential calculus are passed in rapid review, and the notes and examples in small type lighten dark corners in the theory. In section 106 the "theorem of the Mean" is stated and fitly called "the most important theorem in the differential calculus." In section 111, "Integration" is reached and considered as the "converse" of "differentiation"—perhaps the commonsense way of setting about the mattter. In eleven paragraphs of this section, Professor Hardy gives a useful resumé of the possibilities of the solution of the "converse" question.

In chapter VII additional theorems in the calculus such as Taylor's (called the general Mean value theorem) are given with their applications to problems of maxima and minima, contact of curves, etc., the distinguishing feature being that the doctrine of limits, as explained in the previous chapters, is the corner-stone on which each discussion rests.

Chapter VIII deals with the "convergence of infinite series and infinite integrals"—subjects fully treated in Professor Bromwich's book, and the author has purposely avoided the inclusion of anything that really involves difficult ideas—such as "the principle of convergence," "uniform convergence," "double series," etc.

Chapter IX and X, the last two chapters in the book, deal with the theory of the logarithm and the exponential, starting from the definition of a logarithm as an integral. And it is noteworthy that "it was the desire of Professor Hardy to write an elementary account of this theory" that originally led him to begin the book, and the choice of matter for the earlier chapters was decided by a consideration of what theorems would be wanted in the last two. It is usual in English books on the calculus to deduce the differential coefficient of log x from the exponential limit or from the exponential series (*vide* section 198, page 367 of the book), but these chapters contain an independent treatment of the logarithmic function, and this way of introducing the function may appear to be desirable in a first course on the calculus. [A paper by Bradshaw on the subject, in the Annals of Mathematics (2), volume 4, 1903, may be consulted. Also Appendix II, pp. 396–414, of Professor Bromwich's "Infinite series."]

Professor Hardy's book will be eminently useful as a *vade mecum* to those who have already acquired some familiarity with the calculus from other sources. It can hardly be used as a first text-book. The Professor's experience as a teacher of "a good many of the ablest candidates for the Mathematical Tripos" and his own erudition, have enabled him to produce a work which may be deemed "Elementary" only in the sense that it exposes the philosophy of the elements, thereby showing that all beginnings are difficult, and that what lies at the root is always difficult to get at and has required, and will always require, volumes to be written.

It is customary with reviewers to run their critical eye mainly on the defects in a work. This has not been my purpose, the excellence of the work is so marked in many respects. But I may indicate one or two points which may be looked into when another edition of the book is issued, as is sure to be the case. In the seventh line of the note in small type to section 5, page 12, the word "first" should perhaps have been "last." In paragraph 4 of section 112, chapter VI, page 221, it is written "that there really is a function $F(x)$ such that $D_xF(x) = 1/x$ will be *proved* in the next chapter." The reader naturally would search for the proof in one of the sections in large type, but he would fail to find it unless he scans closely the small print of the notes, and arrives at the note at the end of section 142, page 278, which only *indicates* the proof in four brief lines. Section 198, page 367, should, obviously, not have been printed in *small* type.

A. C. Bose.

Traité de Trigonometrie.—Par J. A. Serret.

The scope of the treatise is clearly indicated in the preface in the following words: "The first chapter deals with the elements of the theory of circular functions; the second relates to the construction and use of trigonometrical tables; the two following chapters deal with the subject of trigonometry, properly so-called, *i.e.*, the body of principles on which the solution of triangles, both plane and spherical, depends. These four chapters constitute the elementary portion of the work The fifth chapter completes the theory of circular functions [including the theory of the complex variable and convergency of infinite series and products], of which a sufficiently comprehensive account is given, such as is required in the higher parts of Mathematics. And finally, the sixth and last chapter is specially devoted to the development of trigonometrical solutions based on the employment of series. These solutions relate to the various problems that frequently present themselves in Astronomy and Geodesy, but for which the general methods are found to be insufficient."

Thus the work extends over the whole range of Trigonometry both plane and spherical. It is not without interest, therefore, to compare the mode of presentation of the subject, followed in this work (and in Continental works on mathematics), and that followed in English works generally. In an English text-book the subject-matter of a single chapter of the present work would be dealt with in several short chapters, each one being followed by a large number of illustrative examples (some of them fully worked out). In this work, this long chapter is either followed by no examples, or perhaps six or seven (*questions proposées*) at most

There is an obvious advantage in the English system to the beginner, unless too much time is wasted in solving examples, made artificially difficult as a test of mathematical skill. On the other hand, the mode of presentation followed in this book gives a comprehensive view of the whole subject, as against a somewhat scrappy notion that a student often comes to have by following the English plan. The new regulations of the University of Cambridge are intended to direct attention to the advantage of combining the two systems, and there is no doubt that that is the best course.

D. N. M.

"The Corpuscular Theory of Matter."—By J. J. Thomson.

The celebrated experiments of Hertz has so to say subordinated Optics to Electricity. The modern tendency of physics is to subordinate even the oldest part of physics, namely, Mechanics, to Electricity. We have got to change our views about Matter and its properties and think that the very origin of Mass is probably electrical. Prof. J. J. Thomson is one of the physicists who created this new chapter of physics, and everything whatever comes from his pen should be read by everybody who wants to remain up-to-date in these matters.

The general scientific world had a general view of these new theories in Prof. Thomson's celebrated Silliman lectures. This new book covers the same ground and gives some endeavours to explain some physical properties with the help of the theory of electrons.

In the first chapter we have a short recapitulation of Cathode rays, ideas of electrons, etc. (Those who want detailed information about these, will find them in the excellent book of Schmidt on "Kathod-enstrahlen.")

In the second chapter it is shown that at least a part of the Mass in a charged body has electrical origin. We see also how the Zeeman effect gives a support to the theory of electrons. (For further information we may recommend H. A. Lorentz's lectures on "Visible and Invisible Motions" and J. J. Thomson's "Electricity and Matter.")

In the third chapter we have a theory on the possible origin of Röntgen rays. (We have it a little more elaborately in "Electricity and Matter.")

The next two chapters are devoted to the explanation of metallic conduction with the help of this theory. Here the reader finds a heap of interesting materials such as the theory of radiation, Wien's law, Hall effect, etc.

To quote Thomson's own words, "it seems natural to regard the corpuscle as one of the bricks of which atoms are built up." In the sixth chapter we find how we could think these bricks to be grouped in the atoms of different elements. Periodic law and the law of chemical affinity is also dealt with according to this new theory.

In the last chapter we have an estimate on the number of corpuscles in the different atoms. We find the number of corpuscles is proportional to the atomic weight.

S. K. D.

Notes and News.

Solution of Waring's Problem.—Dr. David Hilbert, one of the most eminent living mathematicians, has recently communicated to the Royal Society of Sciences of Göttingen a paper, entitled "Proof of the representability of integers by means of a fixed number of n^{th} powers of integers," in which he proves the following theorem, first given over a century ago by Sir John Waring who, however, could not prove it: *Every positive integer is representable as the sum of nth powers of positive integers, the number of these integers lying under a limit which depends only on the exponent n and not on the integer represented.* Before Prof. Hilbert undertook the investigation of this theorem, the combined investigations of Liouville, Maillet, Hurwitz, Landau, Schur and others had resulted in establishing the theorem only for the particular values 2, 3, 4, 5, 6, 7, 8, 10 of n.

On a new method of introducing the Elliptic Functions.—At the last Dublin meeting of the British Association, Mr. Robert Russell explained a new method of introducing the elliptic functions. By considering the functions

$$u=\int_{\delta}^{x}\frac{dx}{\sqrt{f(x)}},\qquad v=\int_{\delta}^{y}\frac{dy}{\sqrt{f(y)}}$$

where

$$f(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$$

and δ is one of the roots of $f(x)=0$, he showed by a simple reasoning that the expression

$$\frac{x-y}{(x-\delta)(y-\delta)}$$

was invariant for transformations of the type

$$x=\frac{l\xi+m}{l^1\xi+m^1}$$

and then that a function ϕ existed such that

$$\frac{x-y}{(x-\delta)(y-\delta)}=\frac{\phi(u-v)\phi(u+v)}{\{\phi(u)\}^2\{\phi(v)\}^2}\,.$$

This function ϕ is no other than the ordinary σ-function.

The Fourth International Congress of Mathematicians.—The Fourth International Congress of Mathematicians was held in Rome in April, 1908. The first three Congresses were held respectively in Zurich, Paris, Heidelberg. The following papers were read at the Rome Congress:—

G. DARBOUX.—Infinitesimal Geometry. F. KLEIN.—On the Encyclopædia of Mathematics. A. R. FORSYTH.—On the present condition of partial differential equations of the second order, as regards formal integration. D. HILBERT.—The method of infinite number of independent variables. H. A. LORENTZ.—The partition of energy between ponderable matter and ether. G. MITTAG-LEFFLER.—On the arithmetical representation of general analytical functions of one Complex Variable. S. NEWCOMB.—The theory of the motion of the Moon, its progress and its present state. E. PICARD.—Analysis and its relation to Mathematical Physics. H. POINCARÉ.—Future of Mathematics. VERONESE.—Non-archimedean Geometry.

Professor Volterra opened the proceedings with a discourse on Mathematics in Italy during the second half of the XIXth century.

It has been decided at the fourth Congress that the next Congress will be held in Cambridge in August, 1912.

International Commission for the Teaching of Mathematics.—The following resolution was adopted at the Fourth International Congress of Mathematicians: The Congress having recognised the importance of a comparative examination of the plans and methods of study for the teaching of mathematics in the secondary schools of different nations, entrusted to Messrs. Klein, Greenhill and Fehr the task of constituting an International Commission which will study these questions and will present a complete report to the next Congress at Cambridge.

The Committee consists of—

President:—PROF. FELIX KLEIN, of Göttingen.
Vice-President:—PROF. SIR GEORGE GREENHILL.
General Secretary:—PROF. H. FEHR, of Geneva.

The Ganita-Sara-Sangraha of Mahaviracharya.—Prof. D. E. Smith of New York announced at the Fourth International Congress of Mathematicians that Professor Rangacharya of Madras would shortly publish a translation of the great Ganita of Mahaviracharya which was written at Mysore about the year 850 A.D. Among the interesting features of this work are the summation of series beyond a certain fixed term, a relative error in the value of $\frac{a}{0}$, many problems including the equation

$x-(bx+c\sqrt{x}+a)=0$, and numerous forms of indeterminate equations including $ax+by+cz+dw=p$, $\Sigma a=g$.

Smith's Prizes.—The Smith's Prizes for 1909 have been awarded to Mr. H. W. Turnbull, Trinity College, Cambridge, for his essay, "The irreducible concomitants of two quadratic in n variables"; and to Mr. G. N. Watson, Trinity College, Cambridge, for his essay, "The solution of homogeneous linear difference equation of the second order, and its applications to the theory of linear differential equations of Fuchsian type."

Journal of the Indian Mathematical Club.—We have received the April number of the Journal of the Indian Mathematical Club. It contains the following articles and notes :—

BALAK RAM, M.A., I.C.S.—Common factors of $\dfrac{n!}{m!(n-m)!}$.

R. P. PYRANJPYE, M.A.—An electoral anomaly. H. Cox, M.A.—Self-conjugate triangles. M. J. NARANIENGAR, M.A.—The Euler's Line. K. J. SANJANA, M.A.—On Conics.

Prize Problems.—The following announcements have been made by the Academy of Sciences of Paris :—

The Grand Prize (of 3,000 francs) will be awarded in 1910 for a treatment of the following question: It is known how to find the system of two functions which are meromorphic in the plane of a complex variable and are connected by an algebraical relation. The treatment of a similar question for three uniform functions of *two* complex variables is required.

The Prize Vaillant (of 4,000 francs) will be awarded in 1911 for a work which will render perfect, in some sense, the study of the movement of an ellipsoid in an indefinite liquid, the viscosity of the liquid being taken into account.

The Grand Prize (of 3,000 francs) for 1908 was divided between Prof. Luigi Bianchi of the University of Pisa, and Prof. Guichard of the University of Clermont-Ferrand. The subject was the deformation of the general surface of the second degree.

Wolfskehl Prize.—Dr. Paul Wolfskehl of Dermstadt has left a legacy of 100,000 marks for the foundation of a prize to be awarded to the person who will first prove the great theorem of Fermat, *viz.*, to establish the impossibility of finding any integral solution of the indeterminate equation $x^\lambda+y^\lambda=z^\lambda$ for values of λ greater than 2, where x, y, z are $\neq 0$; or to determine the values of λ for which the equation is valid.

A Treatise on Differential Calculus.—We are glad to announce that Dr. Ganesh Prasad intends to bring out a book on Differential Calculus in two volumes. The first volume, which is intended for beginners, is already in the Press, and is expected to be out by September next. As compared with the current English books on the subject, this volume will be characterized by greater rigour and simplicity of treatment, almost every article being illustrated by numerous worked-out examples.

The second volume, which is in course of preparation, will deal with the advanced portions of the subject. The publishers are Messrs. Longmans, Green and Co., London.

New Publications.

Klein, Wendland, Brandl, and Harnack.—**Universität und Schule.** Lectures delivered at the meeting of German Philologists and Schoolmasters at Bâsle on 25th September, 1907. With an appendix containing the proposals of the Teaching-Committee of all German nature-investigators and physicians with respect to the scientific education of candidates preparing for posts of teachers in Mathematics and Natural Science.

G. Arnoux.—**Arithmétique graphique. Les espaces arithmétiques, leurs transformations.** Pp. 84. 3 francs. Gauthier-Villars, Paris. 1908.

J. G. Leathem.—**The Elementary Theory of the Symmetrical Optical Instrument.** Cambridge Tracts in Mathematics and Mathematical Physics, No. 8. Pp. 74. 2*s*. 6*d*. net. University Press, Cambridge. 1908.

G. F. C. Searle, F.R.S.—**Experimental Elasticity.** A Manual for the Laboratory. Pp. 187. 5*s*. net. University Press, Cambridge. 1908.

R. Baire.—**Leçons sur les Théories générales de l'Analyse.** Tome II: Variables Complexes. Applications géométriques. Pp. 347. 17 fr. Gauthier-Villars, Paris.

H. Burkhardt.—**Vorlesungen über die Elemente der Differential—und Integralrechnung** und ihre Anwendung zur Beschreibung von Naturerscheinungen. Pp. 252. 6 marks. B. G. Teubner, Leipzig.

Arthur Schultze.—**Graphic Algebra.** Pp. 93. 7*s*. 6*d*. Macmillan and Co. London.

H. Fehr.—**Application de la Méthode Vectorielle de Grassmann á la Géométrie infinitésimale.** 2nd Ed. Pp. 94. 4 fr. Georg Co. Geneva.

H. Fehr.—**Enquête de "l'Enseignement Mathématique" sur la Méthode de travail des Mathématiciens.** Pp. 126. 5 fr. Gautheir-Villars, Paris.

E. Fabry.—**Traité de Mathématiques générales** à l'usage des chimistes, physiciens, ingénieurs et des élèves des facultés des sciences, avec une preface de G. Darboux. Pp. 440. 9 fr. A. Hermann, Paris.

J. Edmund Wright.—**Invariants of Quadratic Differential Forms.** Cambridge Tracts in Mathematics and Mathematical Physics, No. 9. Pp. 90. 2*s*. 6*d*. net. University Press, Cambridge. 1908.

F. KLEIN.—**Elementarmathematik vom höhern Standpunkte aus.** Teil I Arithmetik, Algebra, Analysis. Pp. 590. 7 mk. 50. B. G. Teubner, Leipzig.

E. NETTO.—**Gruppen-und Substitutionentheorie** (Sammlung Schubert) Pp. 175. 5 mk. 20. G. J. Goschen, Leipzig.

R. S. BALL.—**Spherical Astronomy.** Pp. 506. 12*s.* net. University Press, Cambridge. 1908.

G. H. HARDY.—**A Course of Pure Mathematics.** Pp. 428. 12*s.* net. University Press, Cambridge. 1908.

H. A. LORENTZ.—**Abhandlungen über Theoretische Physik.** 2 vols. 16 mk. B. G. Teubner, Leipzig.

E. H. ASKWITH.—**The Analytical Geometry of the Conic Sections.** Pp. 443. 7*s.* 6*d.* net. A. and C. Black, London. 1908.

H. ANDOYER.—**Cours d'Astronomie.** Second Part. Pp. 304. 10 fr. A. Hermann, Paris. 1909.

MAXIME BÔCHER.—**An Introduction to the Study of Integral Equations.** Cambridge Tracts in Mathematics and Mathematical Physics. No. 10. Pp. 71. 2*s.* 6*d.* net. University Press, Cambridge. 1909.

E. H. BARTON.—**A Text-book of Sound.** 10*s.* net. Macmillan and Co. London.

ROYAL SOCIETY OF LONDON.—**Catalogue of Scientific Papers.** 1800–1900. Subject index. Volume I. **Pure Mathematics.** 21*s.* net. University Press, Cambridge.

PERCIVAL LOWELL.—**Mars as the Abode of Life.** 10*s.* 6*d.* net. Macmillan and Co. London.

J. LARMOR.—**The Scientific Career of Lord Kelvin.** From the Obituary Notices of the Royal Society. Pp. 76. 4*s.* Harrison and Sons, St. Martin's Lane, London, W.C.

J. L. COOLIDGE.—**Elements of Non-Euclidean Geometry.** University Press, Oxford.

G. E. HALE.—**The Study of Stellar Evolution.** An account of some modern methods of Astrophysical research. Pp. 250. 16*s.* 6*d.* Weslay and Son, London.

H. POINCARÉ.—**La théorie de Maxwell et les oscillations hertziennes. La Télégraphie sans fil.** 3rd Ed. Pp. 114. 2 fr. Gauthier-Villars, Paris.

M. CANTOR.—**Vorlesungen über Geschichte der Mathematik.** Vol. IV. B. G. Teubner, Leipzig.

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Deputy Magistrate, Motihari.

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Executive Engineer, Pilibhit.

B. De, B.A., I.C.S.,
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Beni Prasad Chatterji, M.A.,
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Nares Chandra Ghosh, M.A.,
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Subodh Chandra Gangooly, M.A.,
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S. P. Banerji, M.A.,
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R. C. Guha, M.A., B.L.,
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Abinash Chandra Mookerji, B.E.,
Engineer, Bengal Stone Co.,
100, Clive Street, Calcutta.

The Hon'ble Mr. Justice Lal Mohan Das,
Bhowanipur, Calcutta.

Rai Calica Das Dutta, Bahadur, C.I.E.,
Dewan, Cooch-Behar State.

C. C. Dutta, I.C.S.,
4, Gangadhar Babu's Lane, Calcutta.

List of Books.

Baker	Multiply Periodic Functions.
Jessop	Treatise on the Line Complex.
Bromwich	Theory of Infinite Series.
Carslaw	Fourier's Series and Integrals.
Chrystal	Text-book of Algebra. Parts I and II.
Muir	Theory of Determinants.
Smith	Teaching of Elementary Mathematics.
Wallace	Is Mars Habitable ?
Forsyth	Differential Equations.
Routh	Elementary Rigid Dynamics, 1905, 7th edition.
Edwards	Differential Calculus.
Greenhill	Elliptic Functions.
Cajori	Theory of Equations.
Cajori	History of Mathematics.
Tait	Scientific Papers, 2 volumes.
Sylvester	Mathematical Papers, volume I.
Oppolzer	Traité de la détermination des orbites des comètes et des Planètes.
Darwin	Scientific Papers, volume I (Oceanic Tides).
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Chauvenet	Spherical and Practical Astronomy, 2 volumes.
Cayley	Mathematical Papers, volume I.
Todhunter and Leathem	Spherical Trigonometry.
Lagrange	Mécanique Analytique.
Lord Kelvin	Mathematical and Physical Papers, 3 volumes.
Cayley	Elliptic Functions.
Schiaparelli	Astronomy in the Old Testament.
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S. P. Thomson	Light Visible and Invisible.

Proceedings of the American Association for the Advancement of Science. 1884.

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Kelland and Tait	... Introduction to Quaternions.
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Baker	... Elliptic Functions.
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Tannery and Molk	Éléments de la théorie des fonctions elliptiques. Tomes I to IV.
H. Poincaré	Leçons de Mécanique Céléste. Tome I and tome II, 1$^{\text{re}}$ partie.
Appell	Traité de Mécanique Rationnelle. Tomes I to III.
Darboux	Leçons sur la théorie générale des surfaces et les applications géometriques du Calcul Infinitesimal. Tomes I to IV.
Weber, u-Wellstein	Encyklopädie der Elementar-Mathematik, 3, volumes.
F. Klein	Über Riemanns theorie der Algebraischen Functionen und ihren integrale.
Netto	Vorlesungen über Algebra, 2 volumes.
Fricke and Klein	Vorlesungen über die theorie der Automorphen Functionen, 2 volumes.
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Mathieu	Dynamique Analytique.
Poncelét	Traité des propriétés projectives des figures, 2 volumes.
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Bourdon	Trigonométrie rectiligne et sphérique.
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Serret	Traité de Trigonométrie, 1908.
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British Association Reports, 1869, 1870, 1872, 1873, 1896.

Italian Monographs. 15 volumes.

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"	Funzioni Elliticiche.

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Marcolongo	Teoria matematica dello equilibrio die corpi elastici.
Vivanti	Teoria delle funzioni analitiche.
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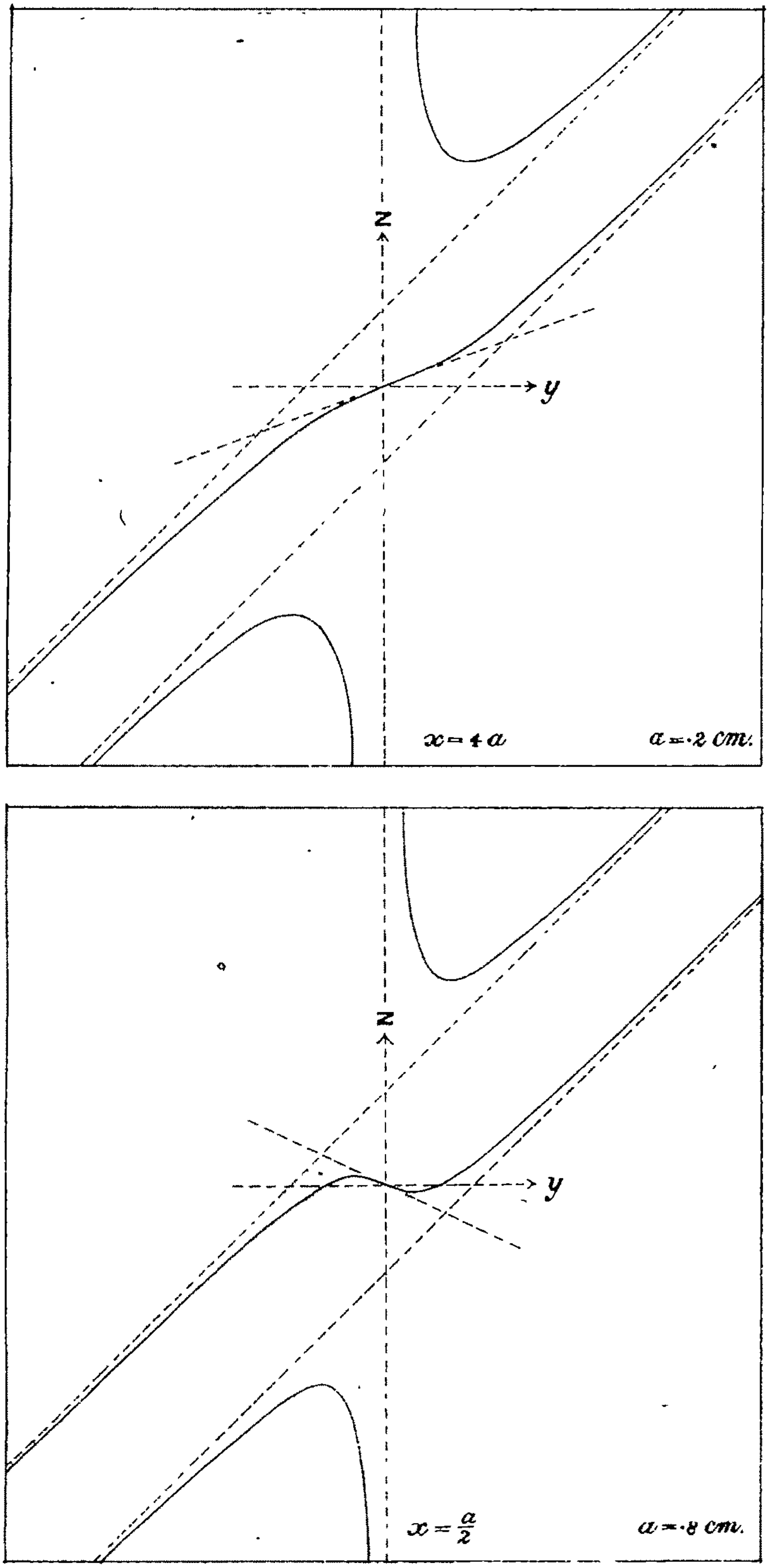
Halsted	Metrical Geometry.
Comstock	Method of Least-Squares.
Manning	Non-Euclidean Geometry.

1.



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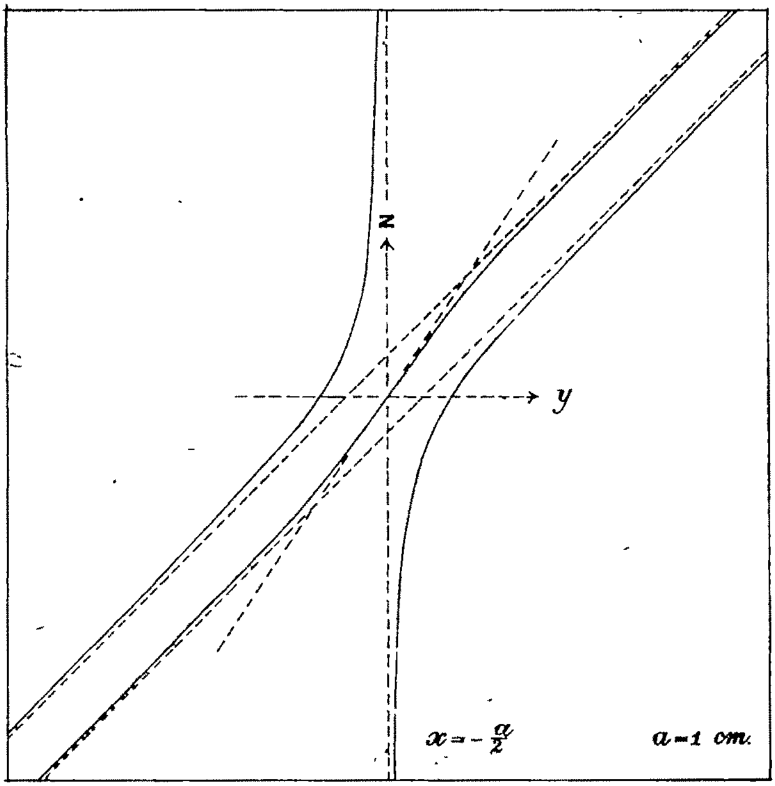
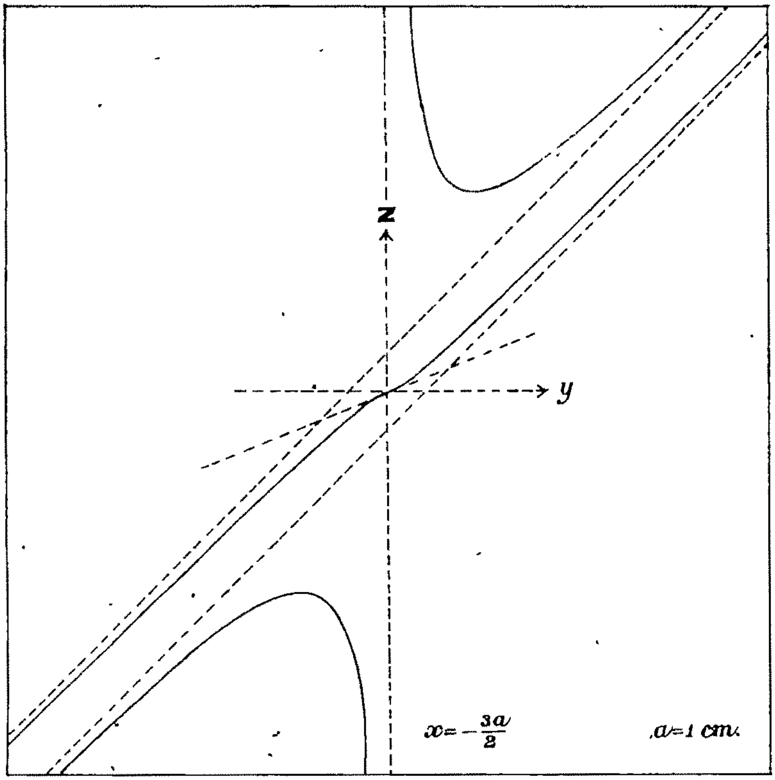
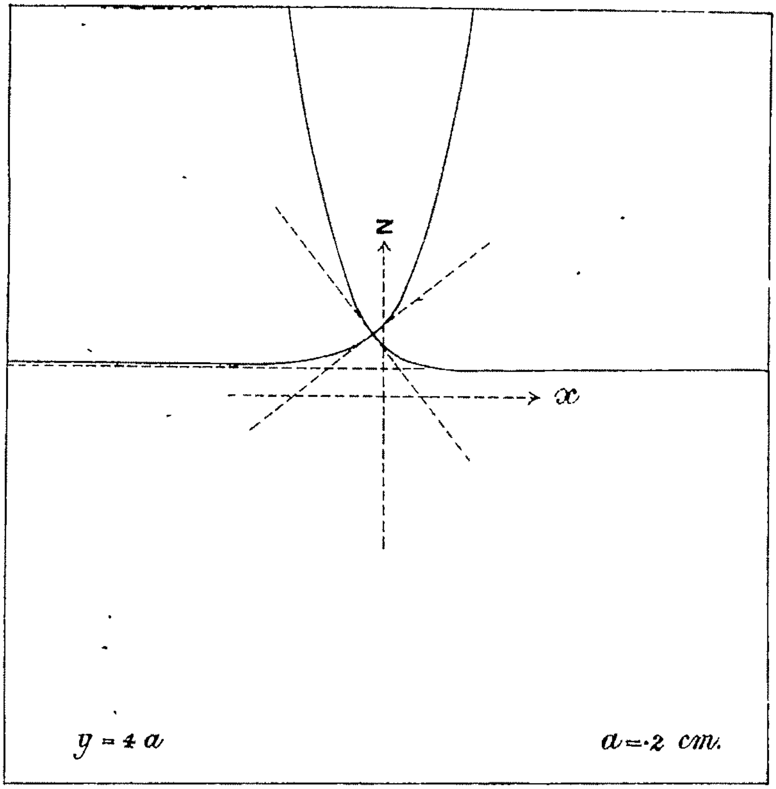
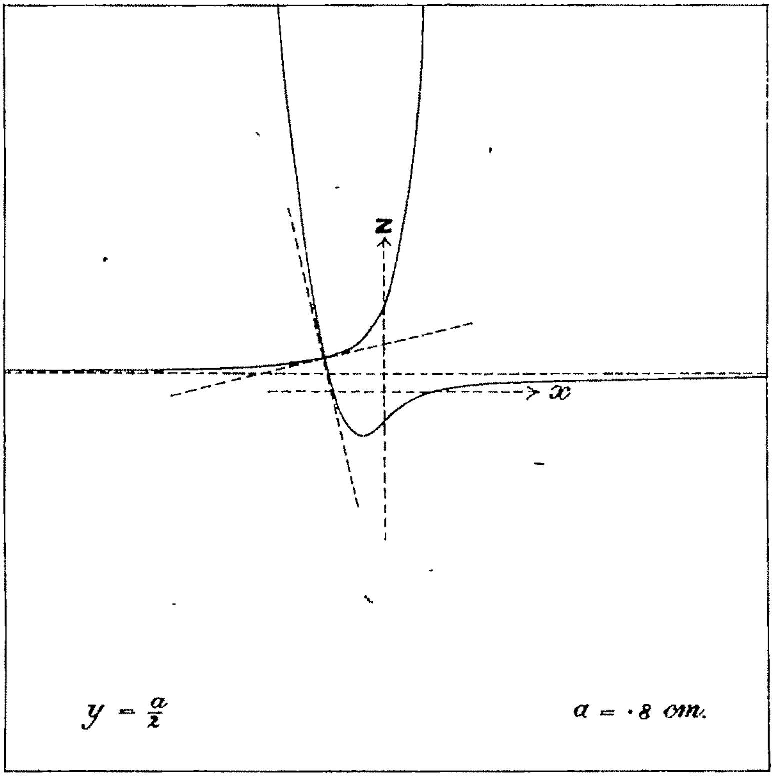
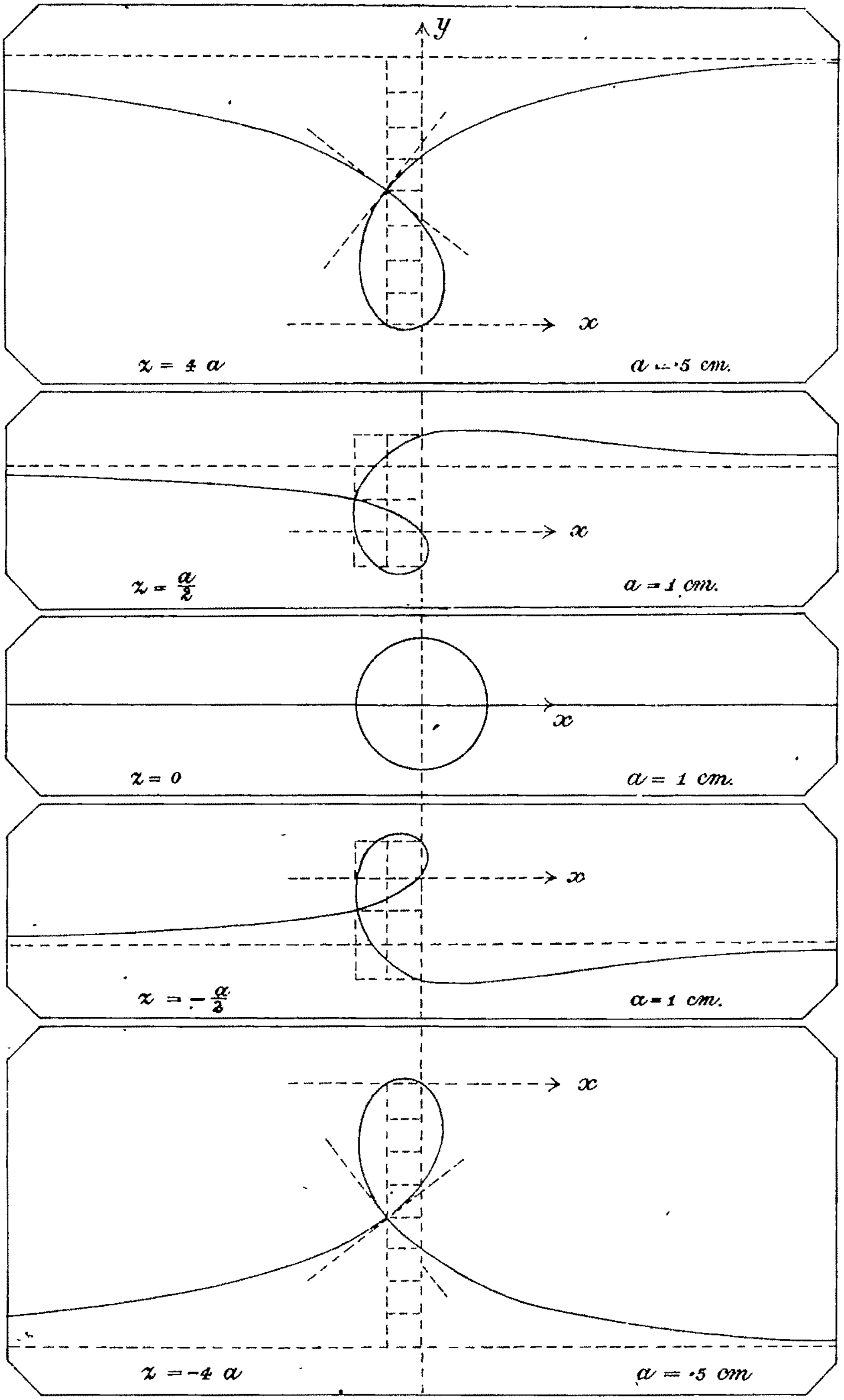





z
x
$y = 4a$
$a = \cdot 2$ cm.


z
x
$y = \frac{a}{2}$
$a = \cdot 8$ cm.

IV



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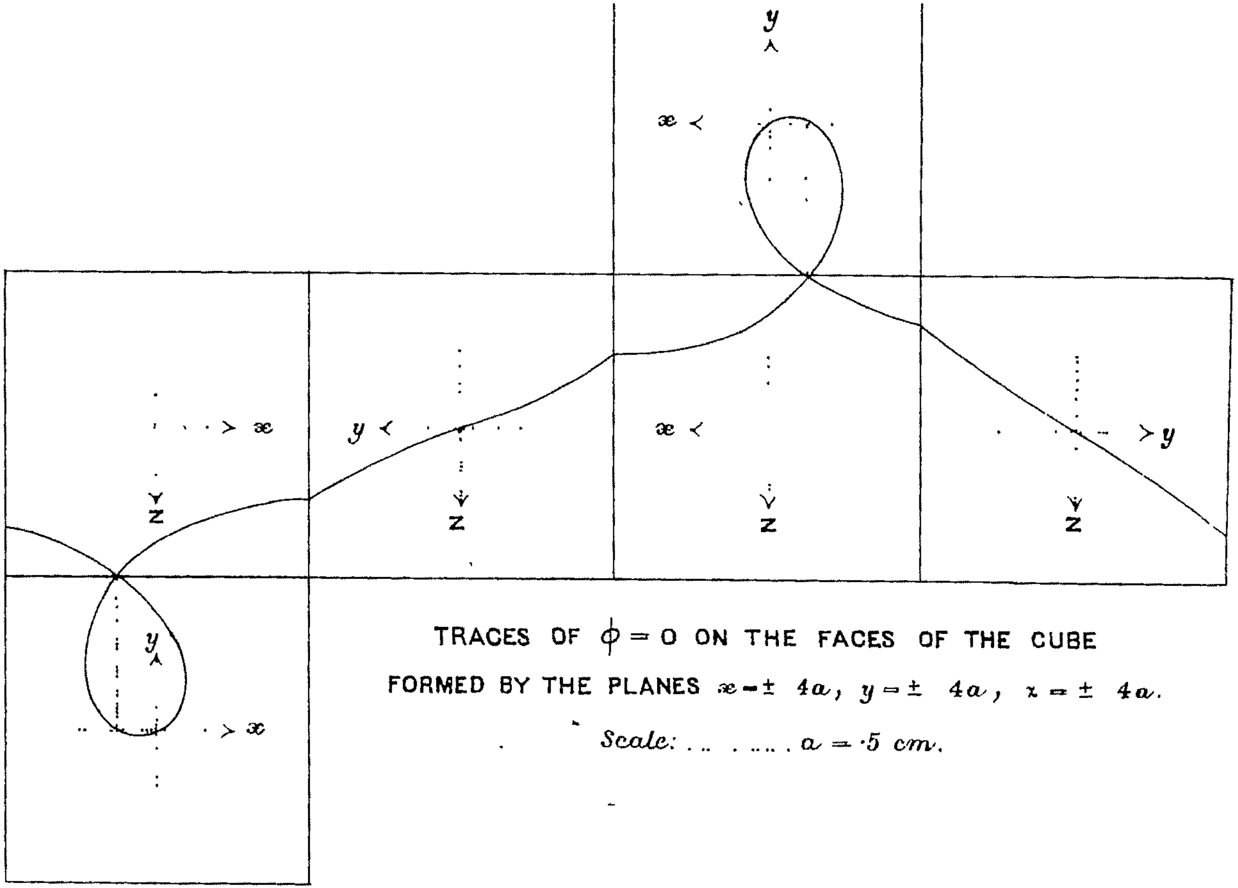
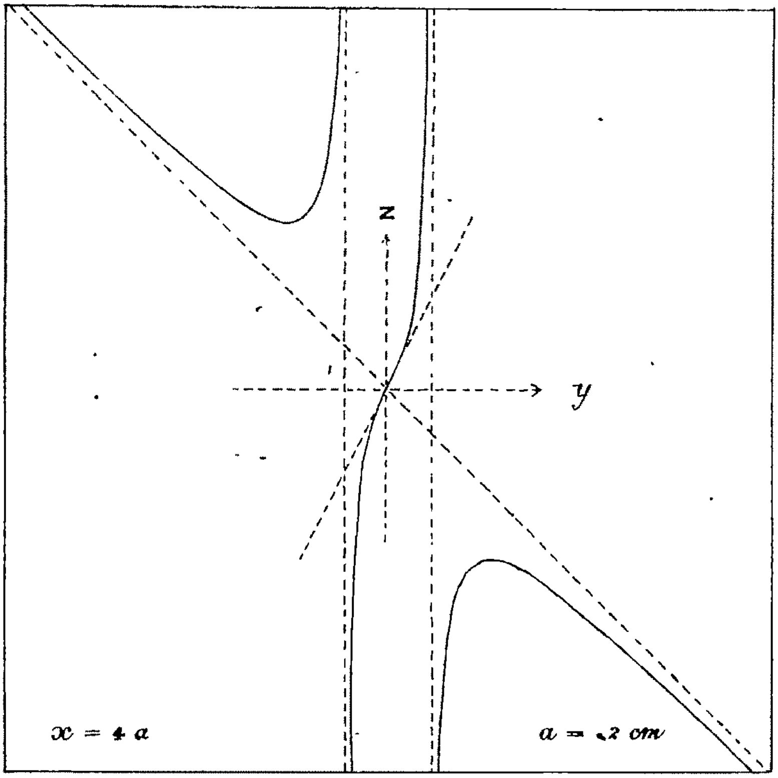


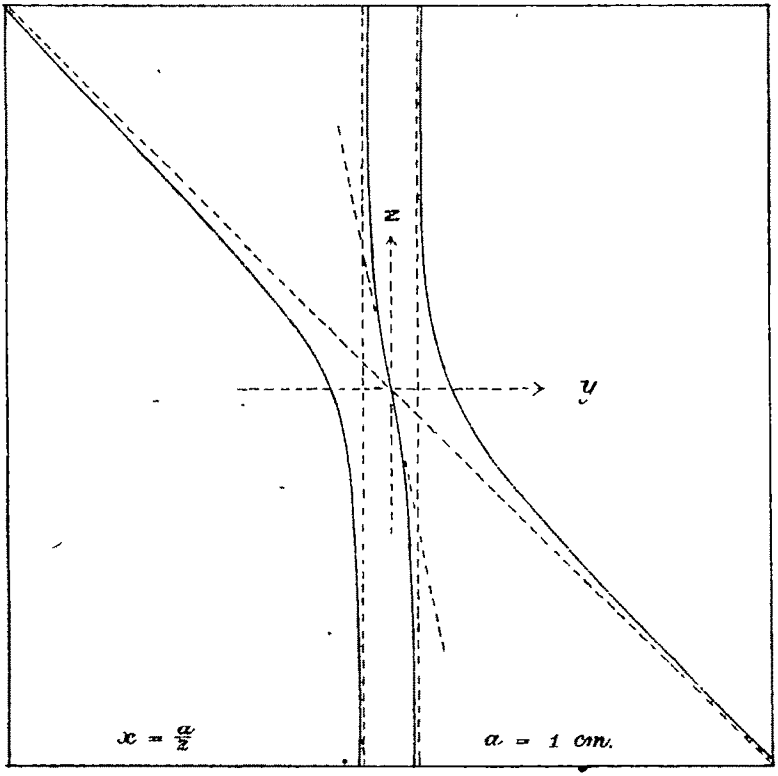
TRACES OF $\phi = 0$ ON THE FACES OF THE CUBE

FORMED BY THE PLANES $x = \pm\ 4a,\ y = \pm\ 4a,\ z = \pm\ 4a.$

Scale: . . . . $a = \cdot 5$ cm.

VI.





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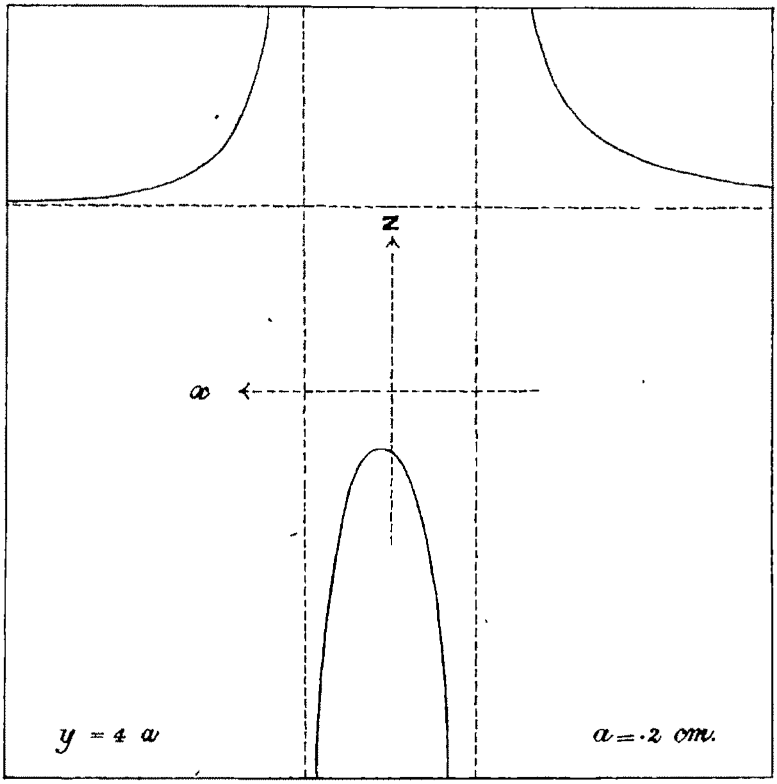

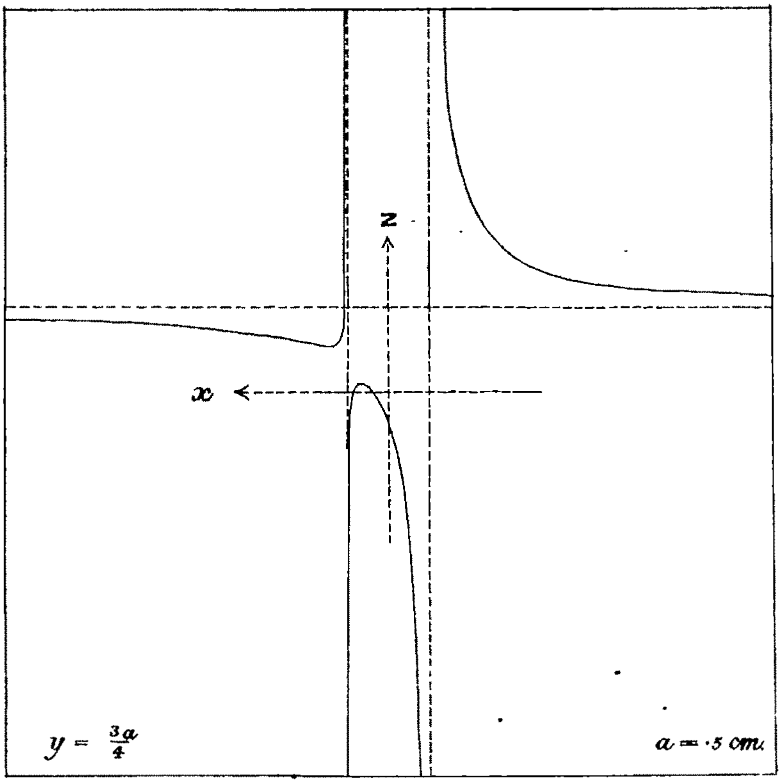

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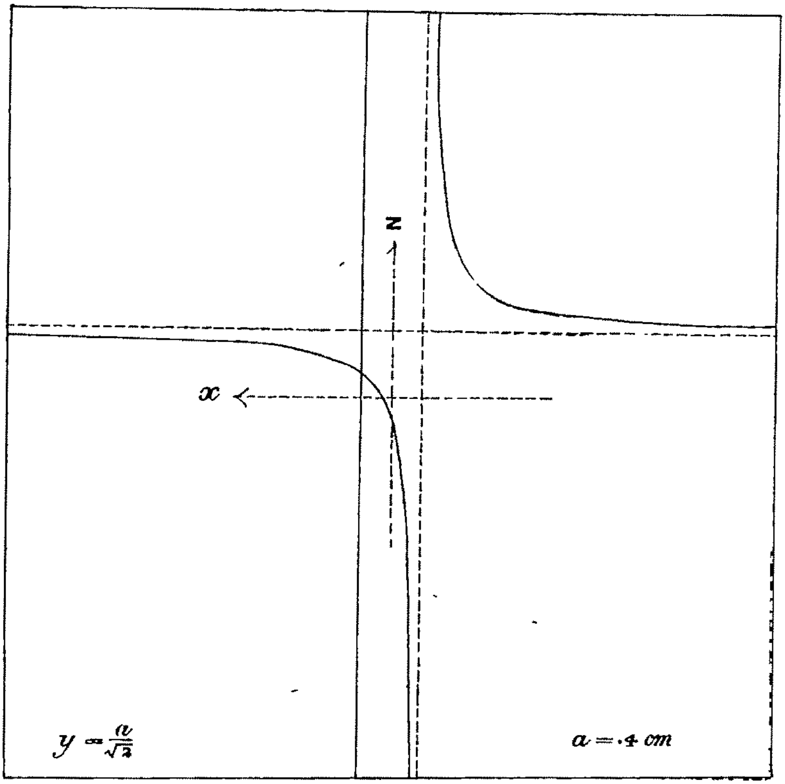
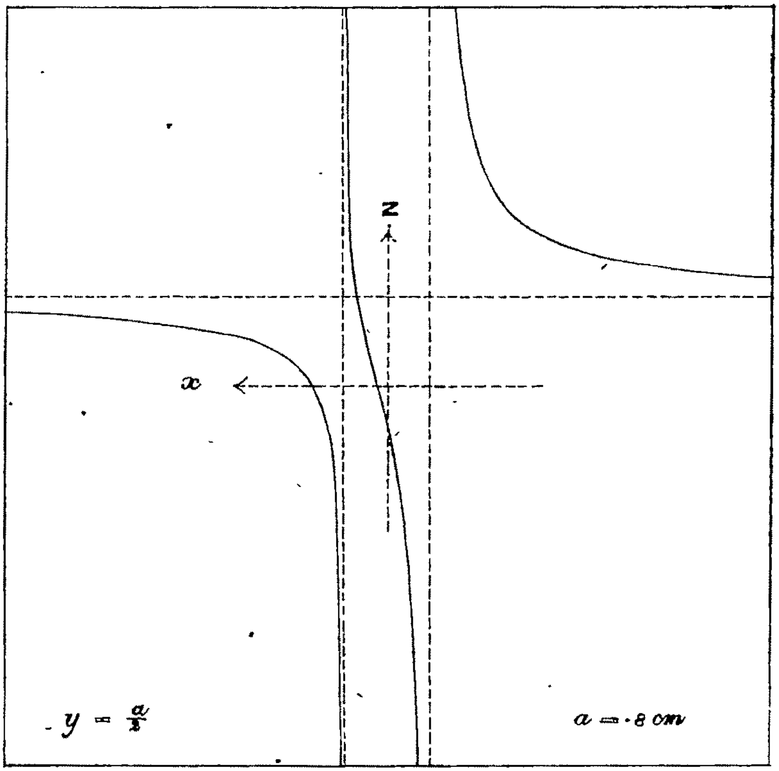
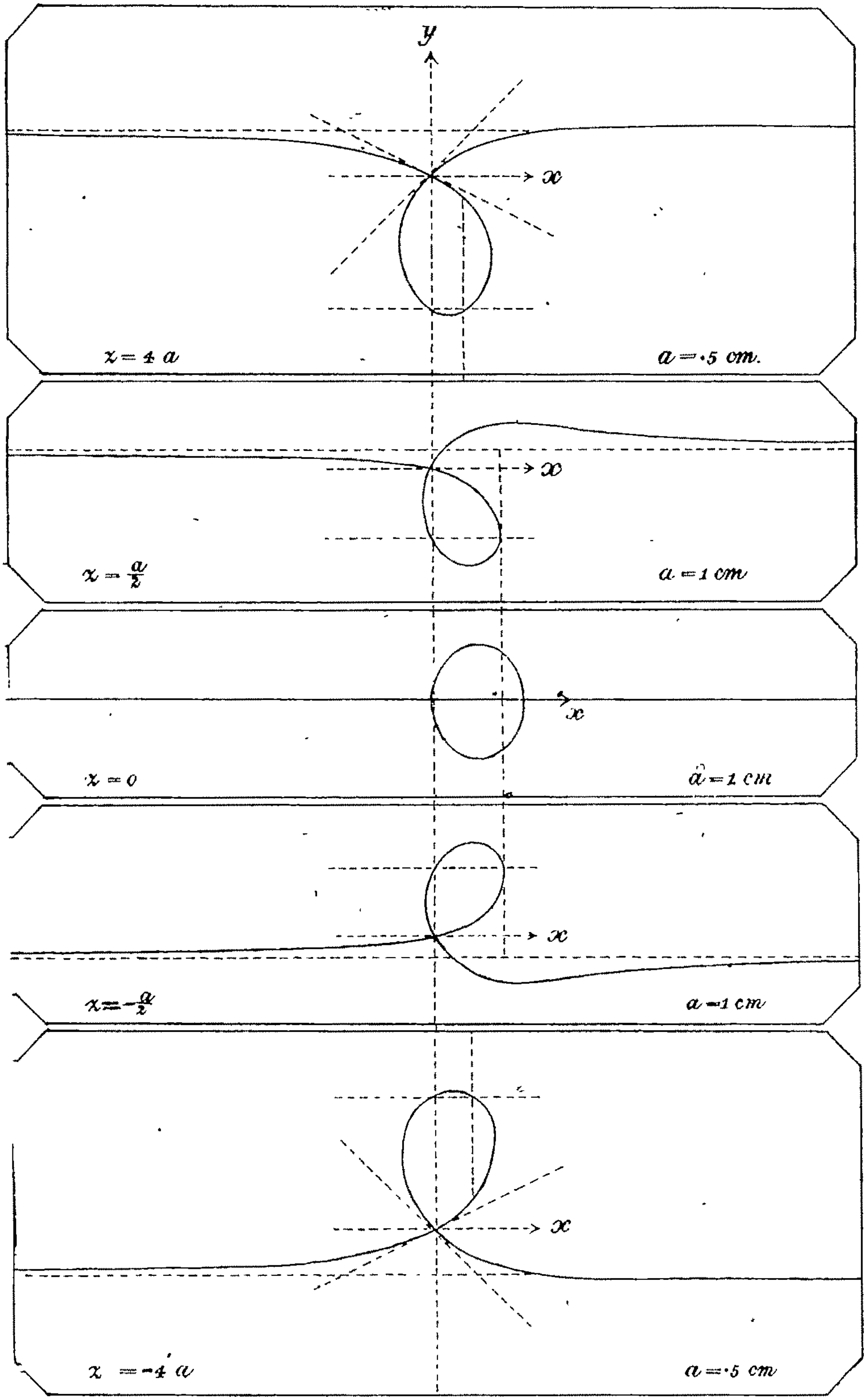
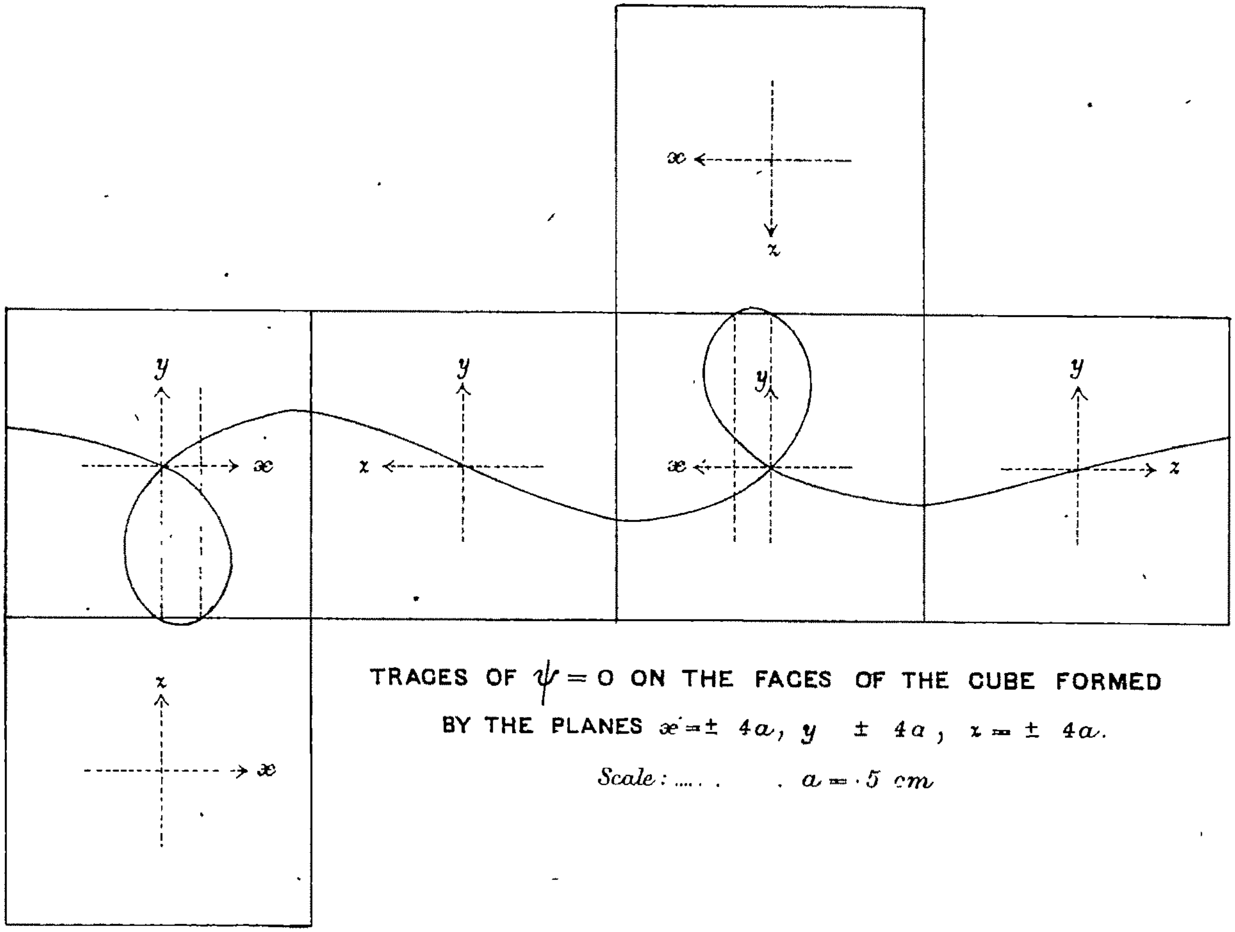




IX


y
x
$z = 4\,a$
$a = \cdot 5$ cm.
x
$z = \frac{a}{2}$
$a = 1$ cm
x
$z = 0$
$a = 1$ cm
x
$z = -\frac{a}{2}$
$a = 1$ cm
x
$z = -4\,a$
$a = \cdot 5$ cm

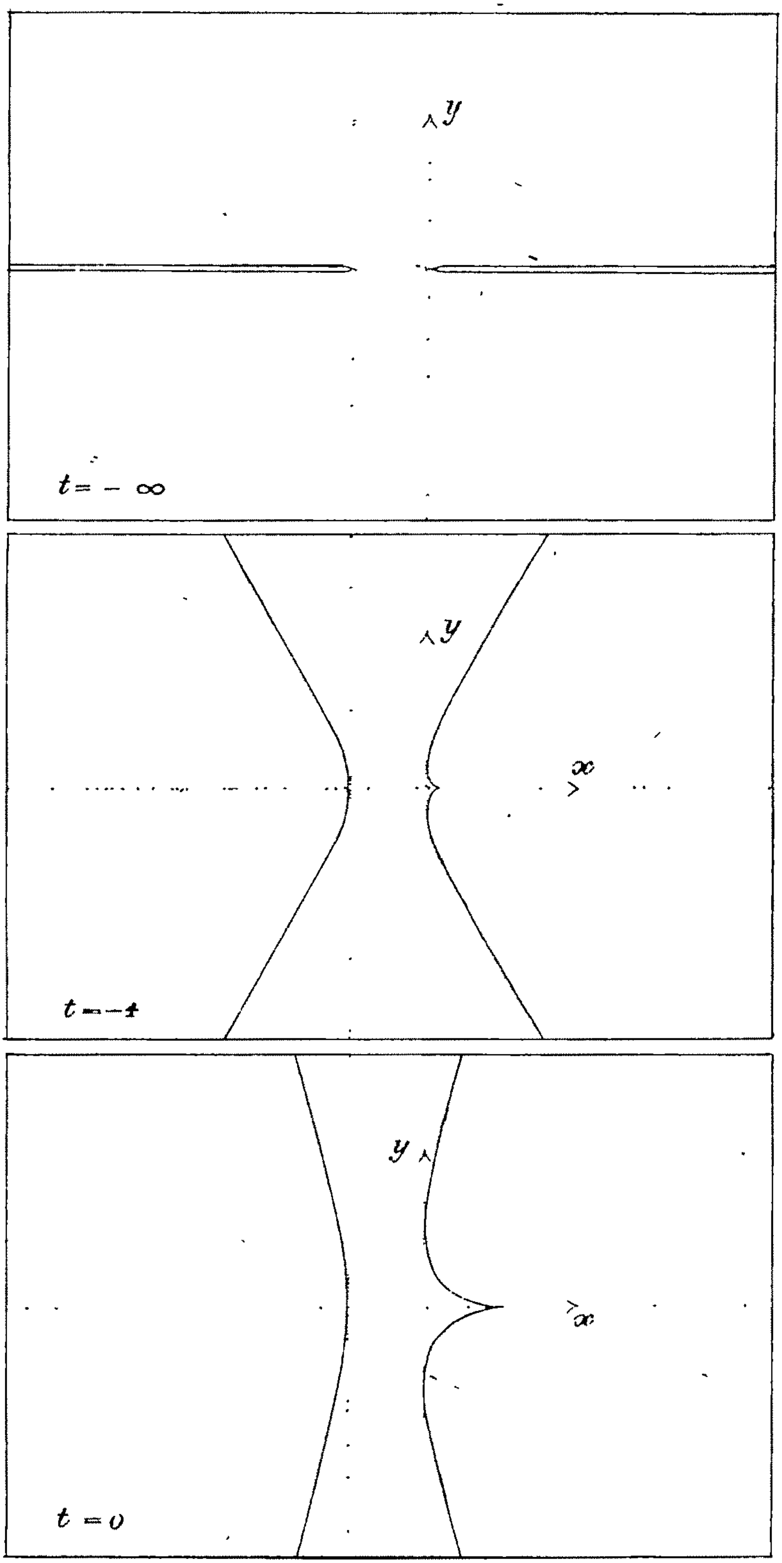
X.



TRACES OF $\psi = 0$ ON THE FACES OF THE CUBE FORMED
BY THE PLANES $x = \pm 4a$, $y \quad \pm 4a$, $z = \pm 4a$.

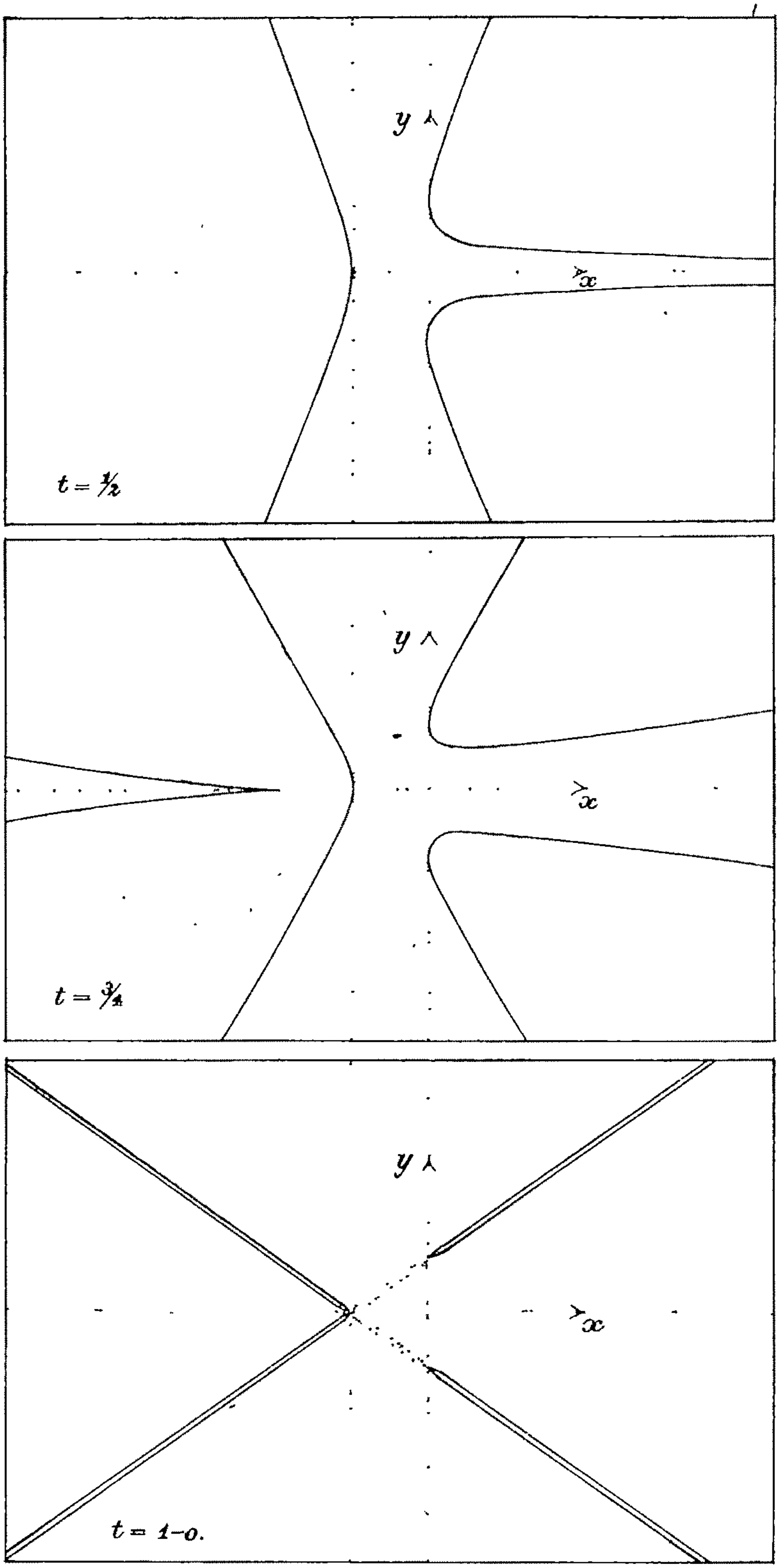
Scale: .... $a = \cdot 5$ cm

XI.



ENVELOPES IN THE PLANES [illegible] TIMES

XII.



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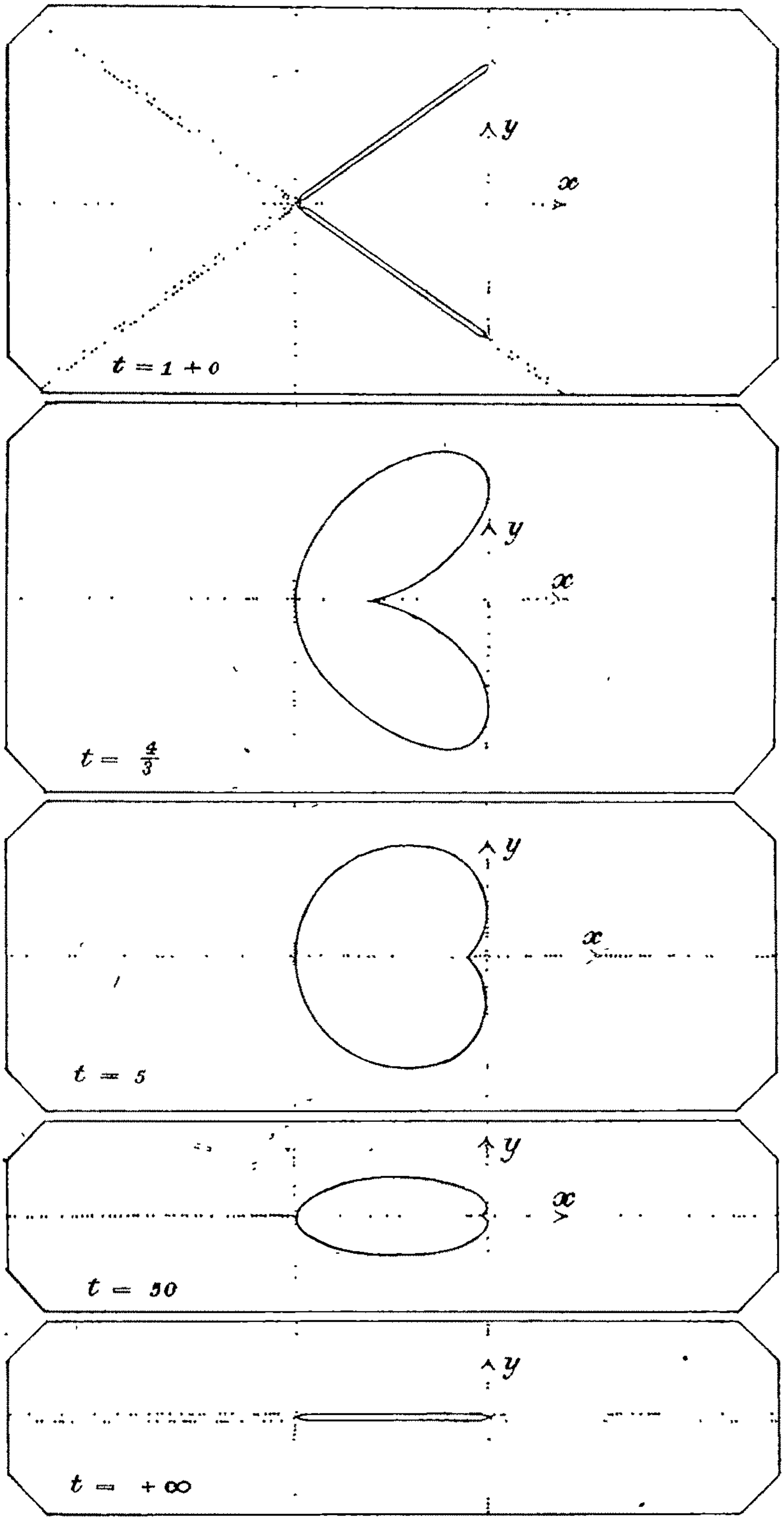
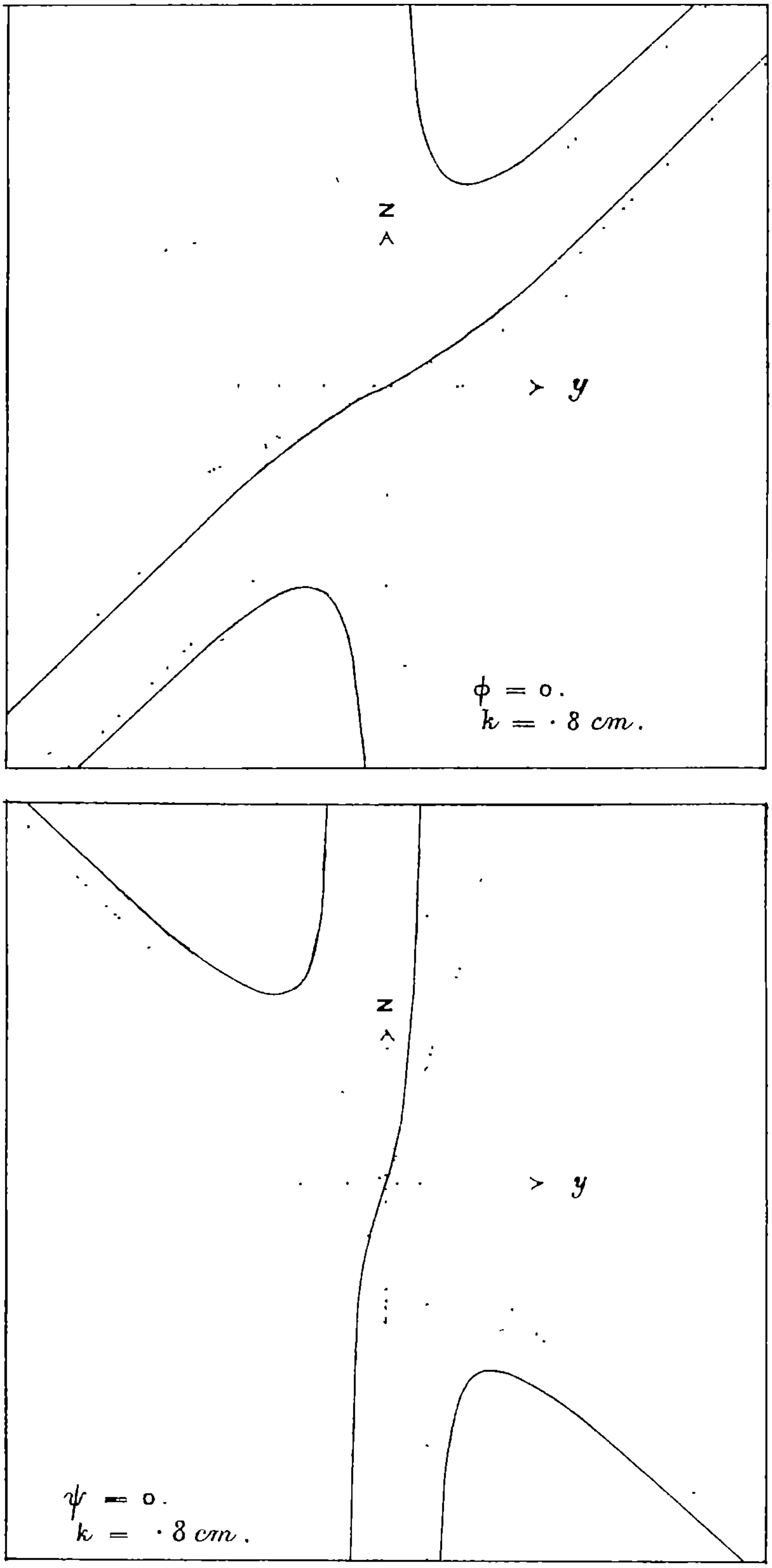


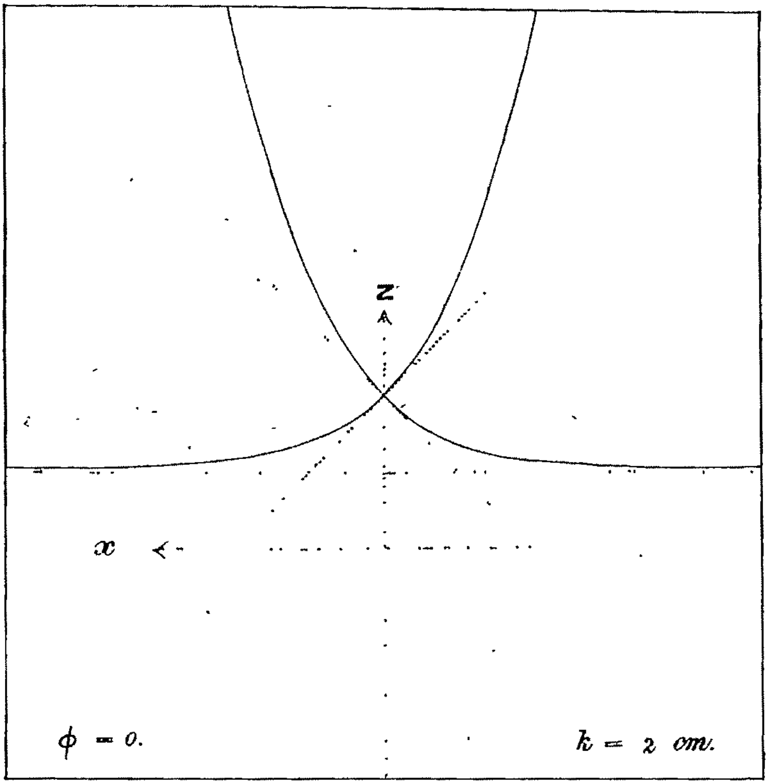
ENVELOPES IN THE PLANES $y + z \tan\alpha = 0$.

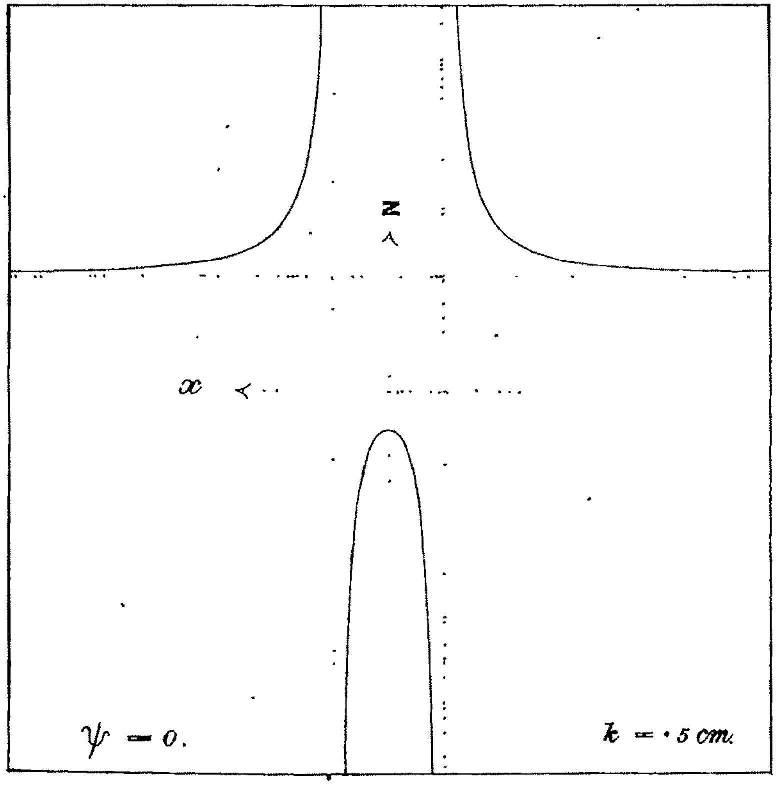
XIV.



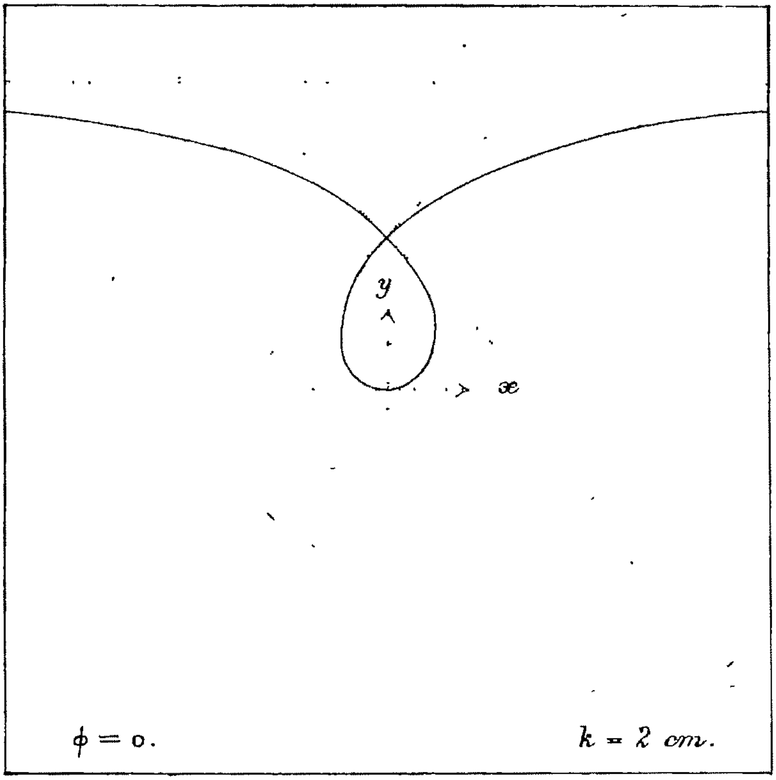
SECTIONS OF THE MÖBIUS CONE BY THE PLANES

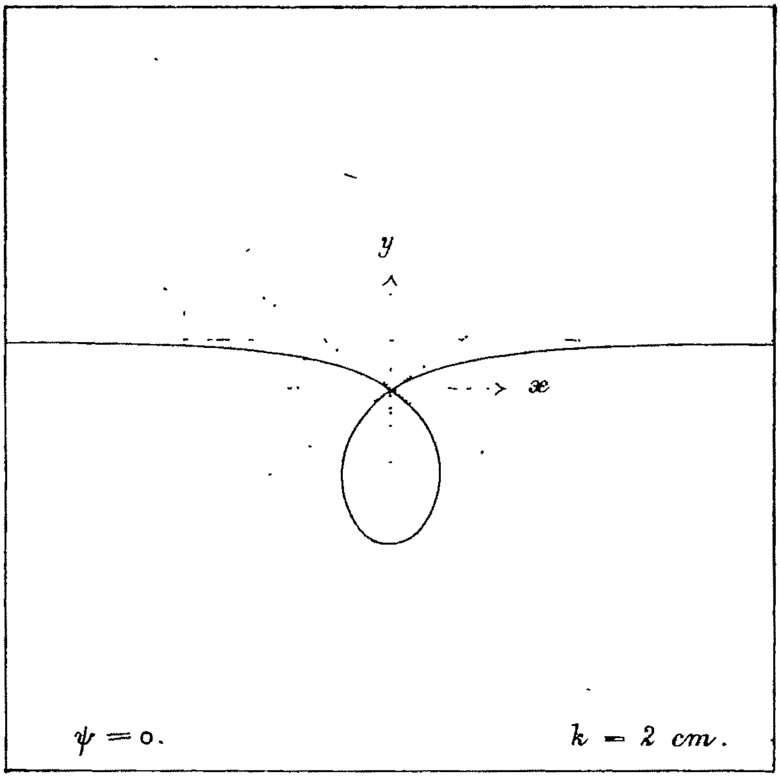
XV.





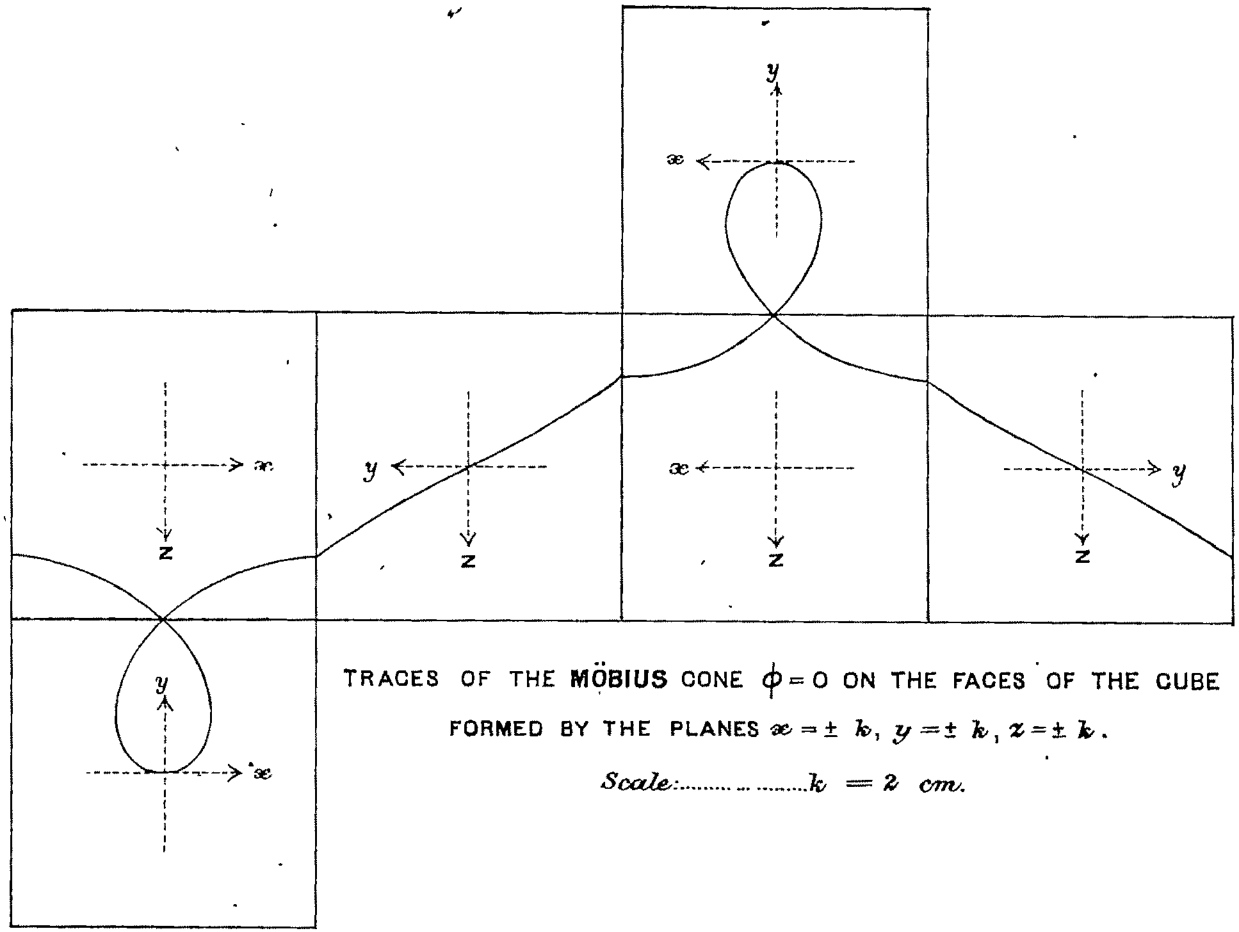
XVI.





SECTIONS OF [illegible]

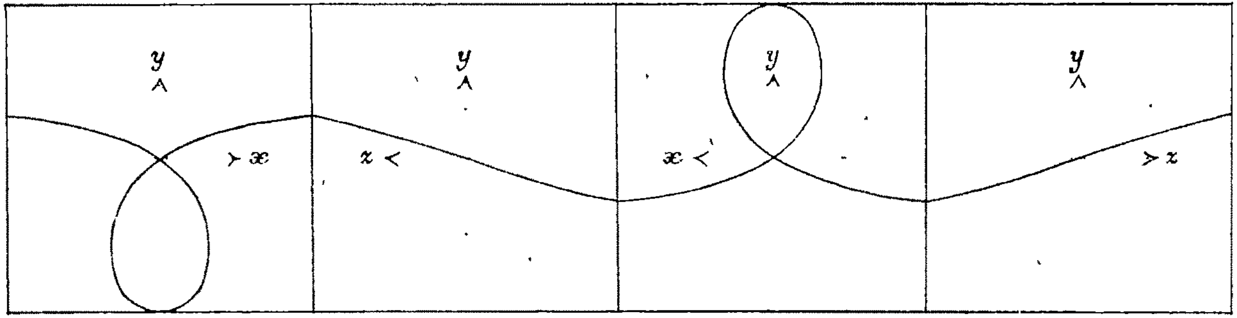
XVII.



TRACES OF THE **MÖBIUS** CONE $\phi = 0$ ON THE FACES OF THE CUBE FORMED BY THE PLANES $x = \pm\, k$, $y = \pm\, k$, $z = \pm\, k$.

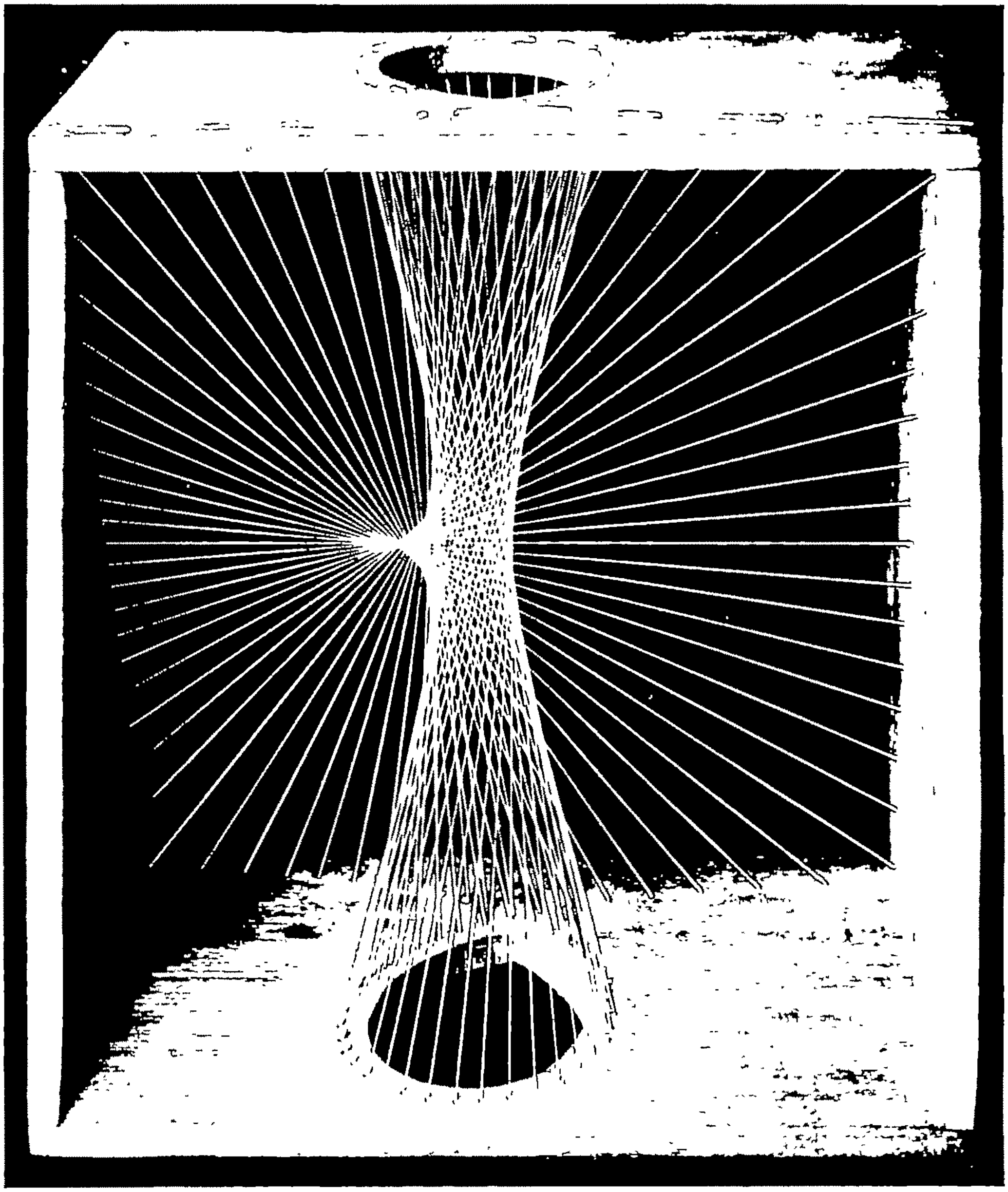
Scale:.....................$k = 2$ *cm.*

XVIII.



TRACES OF THE MÖBIUS CONE $\psi = 0$ ON THE FACES OF THE CUBE
FORMED BY THE PLANES $x = \pm k$, $y = \pm k$, $z = \pm k$.

Scale.............. $k = 2$ *cm.*



MODEL OF THE MÖBIUS SURFACE.

On a Certain Cubic Surface called the Möbius Surface.

(Part II.)

By C. E. Cullis.

13. Envelopes of the Projections of the Generators of $\phi=0$ on the Co-ordinate Planes.

When a model of the surface, constructed in the manner described in § 9, is viewed from any distant position, the envelope of the projections of the generators on a plane perpendicular to the line of sight will be clearly seen, and it will undergo continuous changes in form as the model is turned about, or the position of the observer is changed. The curves which thus present themselves will now be considered.

(1). *Envelope of the projections on the plane* $x=0$.

This plane is parallel both to the direct and to the oblique axis of the surface.

The projection of the generator θ is

$$y=\sin\theta\left(a+z\cot\frac{\theta}{2}\right)$$

or $\qquad yt^2-2at+y-2z=0,$ where $t=\tan\dfrac{\theta}{2}$.

The equation of the envelope, obtained by eliminating t from this equation, and the equation $yt-a=0$ is

$$y^2(y^2-2yz-a^2)=0.$$

It consists of the double line $y^2=0$, and the hyperbola $y(y-2z)=a^2$, whose asymptotes are $y=0$, $y-2z=0$.

(2). *Envelope of the projections on the plane* $y=0$.

This plane is parallel to the direct axis and the initial generator. The projection of the generator θ is

$$x=\cos\theta\left(a+z\cot\frac{\theta}{2}\right),$$

or $$(x+a)t^3+zt^2+(x-a)t-z=0 \qquad (1).$$

To obtain the envelope, t must be eliminated from this equation and
$$3(x+a)t^2+2zt+(x-a)=0 \qquad (2).$$
In virtue of (2) we can replace (1) by $zt^2+2(x-a)t-3z=0$ (3).

The elimination of t from (2) and (3) gives as the equation of the envelope
$$4[z^2-3(x^2-a^2)][3z^2+(x-a)^2]=[9z(x+a)+z(x-a)]^2.$$
If the origin were transferred to the point $(a, 0)$, the equation would become
$$(x^4+11x^2z^2-z^4)+2ax(x^2+18z^2)+27a^2z^2=0.$$
Using the original axes, the leading properties of the curve may be thus described :—

(1) The curve is symmetrical with respect to the x-axis.

(2) There are two real asymptotes, *viz.*, $z=mx+c$,

$$\text{where} \quad m = 3{\cdot}330, \quad \frac{c}{a} = 2{\cdot}070$$

$$\text{or} \quad m = -3{\cdot}330, \quad \frac{c}{a} = -2{\cdot}070.$$

(3) There is a cusp at the point $(a, 0)$, the character of the curve there being $x^3=-\frac{27}{2}az^2$.

(4) There is a vertex at the point $(-a, 0)$, the tangent there being $x=-a$, and the character of the curve being $z^2+8ax=0$.

(5) The line $x=0$ is a double tangent, touching the curve where $z=\pm a$.

(6) A line parallel to the z-axis cuts the curve
in two real points if $x>a$,
in four real points if $x<a$ and >0,
in imaginary points only if $x<0$ and $>-a$,
in two real points if $x<-a$.

Thus the curve lies outside the region lying between the two lines $x=0$, $x=-a$.

(7) The curve consists of two infinite branches, hyperbolic at infinity.

The curve is shown at the foot of Plate XI.*

(3). *Envelope of the projections on the plane* $z=0$.

This consists of the double line $y^2=0$, and a conjugate point at the origin, as is seen by putting $\alpha=90°$ in the next article.

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* The plates referred to in this paper are contained in the preceding number of the *Bulletin*.

14. Envelope of the Projections on the Plane $y+z\tan\alpha=0$.

Those envelopes will now be investigated which are seen when the surface is placed with the initial generator (or x-axis) perpendicular to the line of sight, and is then made to rotate about the initial generator. When the line of sight makes an angle α with the y-axis, $y+z\tan\alpha=0$ will be the plane on which the projections are made.

We introduce a new set of axes defined by the transformation scheme

	x'	y'	z'
x	1	0	0
y	0	$-\sin\alpha$	$\cos\alpha$
z	0	$\cos\alpha$	$\sin\alpha$

The required envelope is that of the projections on the new plane $z=0$.

The equations of the generator θ referred to the new axes are

$$\frac{x-a\cos\theta}{\cos\theta\cos\frac{\theta}{2}}=\frac{z\cos\alpha-y\sin\alpha-a\sin\theta}{\sin\theta\cos\frac{\theta}{2}}=\frac{z\sin\alpha+y\cos\alpha}{\sin\frac{\theta}{2}}.$$

The equation of its projection on the new plane $z=0$ is

$$x\left[\cos\alpha\sin\frac{\theta}{2}-\sin\alpha\sin\theta\cos\frac{\theta}{2}\right]-y\cos\theta\cos\frac{\theta}{2}-a\cos\alpha\cos\theta\sin\frac{\theta}{2}=0$$

or $$t^3.(x+a)\cos\alpha+t^2.y+t.\left[(x-a)\cos\alpha-2x\sin\alpha\right]-y=0$$

where $$t=\tan\frac{\theta}{2}.$$

From this the equation of the envelope will be found to be

$$4\left[3(x+a)(\overline{x-a}\cos\alpha-2x\sin\alpha)\cos\alpha-y^2\right]\left[(\overline{x-a}\cos\alpha-2x\sin\alpha)^2+3y^2\right]$$
$$+z^2.\left[\overline{x-a}\cos\alpha-2x\sin\alpha+9(x+a)\cos\alpha\right]^2=0 \qquad (1).$$

If the origin were transferred to the point $x=\frac{a\cos\alpha}{\cos\alpha-2\sin\alpha}$, $y=0$, the equation would become

$$\left[y^2-x^2.3\cos\alpha(\cos\alpha-2\sin\alpha)-6ax.\cos\alpha(\cos\alpha-\sin\alpha)\right]$$
$$\times\left[3y^2+x^2.(\cos\alpha-2\sin\alpha)^2\right]$$
$$=y^2.\left[x.(5\cos\alpha-\sin\alpha)+\frac{9a\cos\alpha(\cos\alpha-\sin\alpha)}{\cos\alpha-2\sin\alpha}\right]^2 \qquad (2).$$

We can at once describe the following properties of the curve, the axes being those used in the first form of the equation :—

(1) The curve is symmetrical with respect to the x-axis.

(2) The line $x=0$ cuts the curve where $(y^2-a^2\cos^2\alpha)^2=0$, and is therefore a *double tangent.*

(3) The line $x=-a$ cuts the curve where $y=0$, *i.e.*, at a *vertex*, the character of the curve there being

$$y^2+8ax\,.\cos\alpha\,(\cos\alpha-\sin\alpha)=0.$$

(4) There is a *cusp* at the point $x=\dfrac{a\cos\alpha}{\cos\alpha-2\sin\alpha},$ $y=0$, the character of the curve there being

$$x^3=-\frac{27}{2}\,a\,y^2\,.\,\frac{\cos\alpha\,(\cos\alpha-\sin\alpha)}{(\cos\alpha-2\sin\alpha)^4}.$$

The projection of the generator θ touches the envelope where

$$\frac{x}{a}=\frac{1+\cos 2\theta}{(1+2\cos\theta-\cos 2\theta)-2\tan\alpha\,.\,(1+\cos\theta)}$$

$$\frac{y}{a}=(\cos\alpha-\sin\alpha)\,.\,\frac{\sin 2\theta-2\sin\theta}{(1+2\cos\theta-\cos 2\theta)-2\tan\alpha\,.\,(1+\cos\theta)}.$$

We may use these last results to plot the curve for different values of α. In simple cases it may be easier to regard the equation of the envelope as a quadratic in y^2.

15. Asymptotes of the Envelope in the Plane $y+z\tan\alpha=0$.

For points at infinity on the envelope, we have

$$y^4-y^2x^2.\,(11\cos^2\alpha-8\sin\alpha\cos\alpha-\sin^2\alpha)-x^4.\cos\alpha\,(\cos\alpha-2\sin\alpha)^3=0$$

or
$$\frac{y^2}{x^2}=\tfrac{1}{2}\left[P\pm\sqrt{P^2+Q}\right],$$

where
$$P=-\cos^2\alpha\,(\tan\alpha-u)(\tan\alpha-v),$$
$$u=-4-\sqrt{27}=-9{\cdot}196\ldots,$$
$$v=+4-\sqrt{27}=\;\;1{\cdot}196\ldots,$$
$$Q=-32\cos^4\alpha\,(\tan\alpha-\tfrac{1}{2})^3$$
$$P^2+Q=\cos^4\alpha\,(\tan\alpha-1)(\tan\alpha-5)^3.$$

It appears then that

P is positive when $\tan\alpha$ lies between $-4-\sqrt{27}$ and $-4+\sqrt{27}$.

Q is positive when $\tan\alpha<\frac{1}{2}$.

P^2+Q is positive when $\tan\alpha<1$, and when $\tan\alpha>5$.

Hence we obtain the following results :—

If $\tan\alpha>0$ and $<\frac{1}{2}$,

$\dfrac{y^2}{x^2}$ is real, having one positive and one negative value,

$\therefore$ there are 2 *real asymptotes.*

If $\tan\alpha > \frac{1}{2}$ and < 1,

$\frac{y^2}{x^2}$ is real, both values being positive,

∴ there are 4 *real asymptotes.*

If $\tan\alpha > 1$ and < 5,

$\frac{y^2}{x^2}$ is imaginary,

∴ there are *no real asymptotes.*

If $\tan\alpha > 5$,

$\frac{y^2}{x^2}$ is real, both values being negative,

∴ there are *no real asymptotes.*

If $\tan\alpha < 0$,

$\frac{y^2}{x^2}$ is real, having one positive and one negative value,

∴ there are 2 *real asymptotes.*

The forms of the curve corresponding to these various cases are shown in Plates XI, XII and XIII.

16. Range of Real Points of the Envelope in the Plane $y + z\tan\alpha = 0$.

We proceed to show that for real points on the envelope, the limiting values of x for real values of y are $-a$ and 0. Whether y has or has not real values between these limits depends on the value of $\tan\alpha$ being > 1 or < 1. This result, which we shall obtain, can be exhibited in the following tabular form:—

	$x < -a$	x between $-a$ and 0	$x > 0$
$\tan\alpha < 1$	y has real values	y has only imaginary values	y has real values.
$\tan\alpha > 1$	y has only imaginary values	y has real values	y has only imaginary values.

It will be convenient to transfer the origin to the vertex ($x = -a$, $y = 0$). The equation of the envelope then becomes

$$y^4 + y^2\left[(s^2 + 8sc - 11c^2)x^2 - 2(s-c)(s+4c)ax + (s-c)^2a^2\right]$$
$$+ cx\left[(2s-c)x - 2(s-c)a\right]^3 = 0,$$

where s and c are written for $\sin\alpha$ and $\cos\alpha$.

This may be written as

$$y^4+Ay^2+B=0$$

where

$$\begin{aligned} A&=\left[(s+4c)x-(s-c)a\right]^2-27c^2x^2\\ &=\left\{\left[s+(4+\sqrt{27})c\right]x-(s-c)a\right\}\left\{\left[s+(4-\sqrt{27})c\right]x-(s-c)a\right\},\\ B&=cx\left[(2s-c)x-2(s-c)a\right]^3,\\ A^2-4B&=(s-c)(x-a)\left[(s-5c)x-(s-c)a\right]^3. \end{aligned}$$

Now Sturm's functions for the equation $y^4+Ay^2+B=0$ are

$$y^4+Ay^2+B,\quad 2y^3+Ay,\quad -Ay^2-2B,\quad -A(A^2-4B)y,\quad B.$$

For the sake of brevity we will put

$$\begin{aligned} t&=\tan\alpha\\ u&=-4-\sqrt{27}\\ v&=-4+\sqrt{27} \end{aligned}$$

Then the *leading terms* in Sturm's functions for the above equation in y differ by positive factors only from

$$y^4S_1,\ y^3S_2,\ y^2S_3,\ yS_4,\ S_5,\ \text{where}$$

$$\begin{aligned} S_1&=1\\ S_2&=1\\ S_3&=-\left[(t-u)x-(t-1)a\right]\left[(t-v)x-(t-1)a\right]\\ S_4&=-(t-1)(x-a)\left[(t-5)x-(t-1)a\right]\left[(t-u)x-(t-1)a\right]\\ &\qquad\times\left[(t-v)x-(t-1)a\right]\\ S_5&=x\left[(t-\tfrac{1}{2})x-(t-1)a\right]. \end{aligned}$$

The nature of the four roots of the equation in y can be determined by a consideration of the signs of S_5, S_4, and S_3.

If S_5 is $-$, two roots are real and two are imaginary.

If S_5 is $+$,

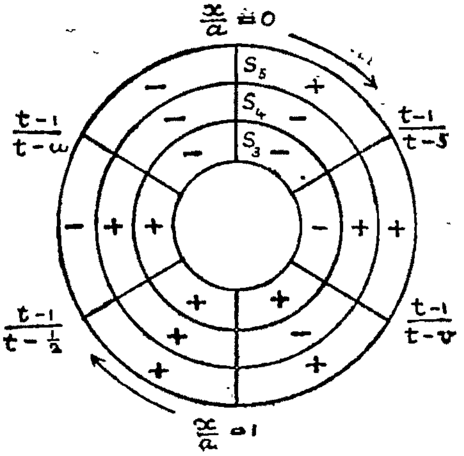
then if S_4 and S_3 are both $+$, all four roots are real;

and if S_4 and S_3 are not both $+$, all four roots are imaginary.

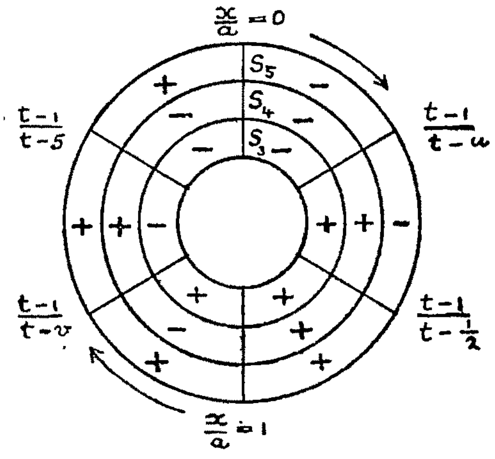
There is one critical value of t, viz, $t=1$, for which changes of sign can occur independently of the value of x.

The critical values of $\frac{x}{a}$ for which the functions can change sign for a given value of t are $0,\ \frac{t-1}{t-5},\ \frac{t-1}{t-v},\ 1,\ \frac{t-1}{t-\frac{1}{2}},\ \frac{t-1}{t-u}$. These values are here written in the order in which the co-ordinate x passes through them as it travels continuously in the same sense from 0 to ∞ and then through ∞ and back to 0, that sense being the positive sense when $t-1$ is $+$, and the negative sense when $t-1$ is $-$. We do not include ∞ among the critical values of $\frac{x}{a}$ because the functions S_5, S_4, S_3 are all of even degree in x.

If now the signs of S_5, S_4, S_3 are determined for any one pair of values of t and x, for example when t and x are very small positive quantities, we can at once deduce their signs in all cases. These signs are exhibited in the annexed diagrams, where the signs of S_3, S_4, S_5 are shown in the inner, middle, and outer ring, respectively.



Case I. $t < 1$.
The arrow shows the direction in which x increases.



Case II. $t > 1$.
The arrow shows the direction in which x increases.

From the first diagram we see that if $t<1$, then as $\frac{x}{a}$ changes continuously in the positive sense from $\frac{t-1}{t-5}$ to $\frac{t-1}{t-v}$, S_5 is always $+$, S_4 always $+$, and S_3 always $-$, and so for the other ranges.

The diagrams show that all four values of y are imaginary as x describes the finite range from 0 to a in Case I, and as x describes the infinite range from 0 to a in Case II, and that for all other values of x, the equation in y has real solutions. Since we have transferred the origin to $(-a, 0)$, this result agrees with that stated at the commencement of this article. The diagrams, however, enable us to describe the natures of the roots in greater detail. In Case I, as x varies in the positive sense from 0 to a, the roots are all imaginary; as it varies from a to $\frac{t-1}{t-\frac{1}{2}}a$ they are all real; as it continues to vary from $\frac{t-1}{t-\frac{1}{2}}\,a$ to 0, they are two real and two imaginary. In Case II, as x changes in the negative sense from 0 to a, the roots are all imaginary; as it changes from a to $\frac{t-1}{t-\frac{1}{2}}\,a$, they are all real; as it changes from $\frac{t-1}{t-\frac{1}{2}}\,a$ to 0, they are two real and two imaginary.

Returning then to the original axes, our complete results are

Case I.

$\tan a < 1$

x varies in the $+$ sense

Range of x	Nature of roots
$-a$ to 0	4 imaginary
0 to $\frac{a}{1-2\tan a}$	4 real
$\frac{a}{1-2\tan a}$ to 0	2 real and 2 imaginary

Case II.

$\tan a > 1$

x varies in the $-$ sense

Range of x	Nature of roots
$-a$ to 0	4 imaginary
0 to $\frac{a}{1-2\tan a}$	4 real
$\frac{a}{1-2\tan a}$ to 0	2 real and 2 imaginary

If we speak of that portion of x space in which y has no real values as the *hiatus*, then in travelling continuously in the same sense along the x-axis we shall pass successively through the *vertex*, *hiatus*, *double tangent*, *cusp*, *vertex*, the sense in which we travel being positive when $\tan a < 1$ and negative when $\tan a > 1$.

These results are illustrated by the figures in Plates XI, XII, and XIII.

17. Line of Striction of the Möbius Surface.

The equation of the surface will be taken in the form

$$\phi(x, y, z) \equiv y(x^2+y^2+z^2-a^2)-2z(x^2+y^2+ax)=0,$$

and it will be shown that the line of striction is the ellipse in which the surface is cut by the plane $2y+z=0$.*

Let (l, m, n) be the direction cosines of the generator θ;

(λ, μ, ν) those of the common perpendicular to the two consecutive generators θ, $\theta+d\theta$;

(p, q, r) the direction-cosines of the normal to the plane through the common perpendicular and the generator θ;

(p', q', r') those of the normal to the plane through the common perpendicular and the generator $\theta+d\theta$.

Then the co-ordinates (x, y, z) of the point in which the generator θ is cut by the common perpendicular to it and the consecutive generator $\theta+d\theta$, satisfy the equations

$$(x-a\cos\theta)p+(y-a\sin\theta)q+zr=0, \qquad (1)$$

$$(x-a\cos\theta+a\sin\theta\, d\theta)p'+(y-a\sin\theta-a\cos\theta\, d\theta)\,q'+zr'=0, \qquad (2)$$

these being respectively the equations of the plane through the common perpendicular and the generator θ, and the plane through the common perpendicular and the generator $\theta+d\theta$.

Now $\quad l=\cos\theta\cos\dfrac{\theta}{2},\ m=\sin\theta\cos\dfrac{\theta}{2},\ n=\sin\dfrac{\theta}{2},$

and
$$\frac{\lambda}{m\dfrac{dn}{d\theta}-n\dfrac{dm}{d\theta}}=\frac{\mu}{n\dfrac{dl}{d\theta}-l\dfrac{dn}{d\theta}}=\frac{\nu}{l\dfrac{dm}{d\theta}-m\dfrac{dl}{d\theta}}=\frac{1}{k},$$

where
$$k^2=\left(\frac{dl}{d\theta}\right)^2+\left(\frac{dm}{d\theta}\right)^2+\left(\frac{dn}{d\theta}\right)^2=\frac{3+2\cos\theta}{4}.$$

Again
$$p=\mu n-\nu m,\ q=\nu l-\lambda n,\ r=\lambda m-\mu l,$$

$$\therefore\quad p=\frac{1}{k}\frac{dl}{d\theta},\ q=\frac{1}{k}\frac{dm}{d\theta},\ r=\frac{1}{k}\frac{dn}{d\theta}.$$

Further
$$p'=\mu(n+dn)-\nu(m+dm),\ q'=(l+dl)-\lambda(n+dn),$$
$$r'=\lambda(m+dm)-\mu(l+dl).$$

$$\therefore\quad p'=\frac{1}{k}\frac{dl}{d\theta}-kl\,.\,d\theta,\ q'=\frac{1}{k}\frac{dm}{d\theta}-km\,.\,d\theta,\ r'=\frac{1}{k}\frac{dn}{d\theta}-kn\,.\,d\theta.$$

---

* The result that the line of striction lies in the plane $2y+z=0$ is contained as an example in *Frost's Solid Geometry*.

∴ equations (1) and (2) are equivalent to

$$(x-a\cos\theta)\,\frac{dl}{d\theta}+(y-a\sin\theta)\frac{dm}{d\theta}+z\frac{dn}{d\theta}=0,$$

$$(x-a\cos\theta+a\sin\theta\,.\,d\theta)\left(\frac{dl}{d\theta}-k^2l\,.\,d\theta\right)+(y-a\sin\theta-a\cos\theta\,.\,d\theta)$$

$$\times\left(\frac{dm}{d\theta}-k^2m\,.\,d\theta\right)+z\left(\frac{dn}{d\theta}-k^2n\;d\theta\right)=0$$

or to

$$\left.\begin{aligned}&(x-a\cos\theta)\,\frac{dl}{d\theta}+(y-a\sin\theta)\,\frac{dm}{d\theta}+z\frac{dn}{d\theta}=0\\ &(x-a\cos\theta)\,l+(y-a\sin\theta)m+zn-\frac{a}{k^2}\left(\sin\theta\,.\frac{dl}{d\theta}-\cos\theta\,.\,\frac{dm}{d\theta}\right)=0.\end{aligned}\right\}$$

Thus the co-ordinates (x, y, z) of the point in which the line of striction cuts the generator θ are given by

$$\frac{x-a\cos\theta}{l}=\frac{y-a\sin\theta}{m}=\frac{z}{n}$$

$$(x-a\cos\theta)\,l+(y-a\sin\theta)\,m+zn-\frac{a}{k^2}\left(\sin\theta\frac{dl}{d\theta}-\cos\theta\frac{dm}{d\theta}\right)=0.$$

From these three equations, it follows that

$$x=\frac{a\cos\theta}{3+2\cos\theta},\quad y=\frac{a\sin\theta}{3+2\cos\theta},\quad z=-\frac{2\,a\sin\theta}{3+2\cos\theta},$$

and therefore that $2y+z=0$.

Hence all points on the line of striction satisfy the two equations:

$$\left.\begin{aligned}2y+z=0.\\ y(x^2+y^2+z^2-a^2)-2z(x^2+y^2+ax)=0.\end{aligned}\right\}$$

Since the plane $2y+z=0$ passes through the initial generator, the line of striction is the conic in which the Möbius Surface is cut by the plane $2y+z=0$.

By § 8, the equation of this conic referred to axes in its plane is

$$5x^2+\tfrac{9}{5}\,y^2+4ax-a^2=0$$

$$\text{or }\frac{25}{9a^2}\left(x+\frac{2a}{5}\right)^2+\frac{y^2}{a^2}=1.$$

Referred to the original axes of the surface, it is an ellipse whose centre is at the point $\left(-\frac{2a}{5},\,0,\,0\right)$ on the initial generator. The major semi-axis is perpendicular to the initial generator and of length a; the minor semi-axes is parallel to the initial generator and of length $\frac{3a}{5}$.

18. Curvature of the Möbius Surface.

Let u be the distance of any point on the generator θ from the foot of that generator, where it intersects the guiding circle, and let (u, θ) be regarded as the co-ordinates of any point on the surface. Then it will be found that the principal radii of curvature are the roots of the following equation in ρ:

$$a^2\rho^2 - 4\rho \sin\frac{\theta}{2} \cdot \left[\tfrac{1}{4}u^2 + \left(a + u\cos\frac{\theta}{2}\right)^2\right]\left[\tfrac{1}{4}u^2 \sin^2\frac{\theta}{2} + \left(a + u\cos\frac{\theta}{2}\right)^2\right]$$
$$-4\left[\tfrac{1}{4}u^2 \sin^2\frac{\theta}{2} + \left(a + u\cos\frac{\theta}{2}\right)\right]^2 = 0.$$

The differential equation of the lines of curvature is

$$adu^2 + 2\sin\frac{\theta}{2}\left[\tfrac{1}{2}u^2 + \left(a + u\cos\frac{\theta}{2}\right)^2\right] du\, d\theta - \left[\tfrac{1}{4}u^2 + \left(a + u\cos\frac{\theta}{2}\right)^2\right]$$
$$\times\, a d\theta^2 = 0.$$

The curves of inflexion are

$$d\theta = 0$$

$$adu + \left[\tfrac{1}{2}u^2 + \left(a + u\cos\frac{\theta}{2}\right)^2\right] \sin\frac{\theta}{2} \,.\, d\theta = 0.$$

The infinitesimal arc of any curve drawn on the surface is given by

$$ds^2 = du^2 + \left[\tfrac{1}{4}u^2 + \left(a + u\cos\frac{\theta}{2}\right)^2\right] d\theta^2.$$

The curve $u = c$, where c is constant and $< a$, is a re-entrant curve traced on a tore, and can be regarded as the line-boundary of a Möbius Ring (see § 1).

19. The Möbius Cone.

When the radius of the guiding circle becomes infinitely small, the Möbius Surface becomes the cone whose equation is

$$\phi \equiv (y - 2z)(x^2 + y^2) + yz^2 = 0 \quad \text{or} \tag{1}$$

$$\psi \equiv x^2(3y - z) + 2y^2(y + z) = 0 \tag{2}$$

according as the first or second set of axes is employed.

The sections of (1) by planes parallel to the co-ordinate planes, and by any plane, are as follows:—

Section by $x = k$.

$y(y - z)^2 + k^2(y - 2z) = 0.$

The *asymptotes* are $y = 0$, $y - z = \pm k$.

The curve is *symmetrical about the origin.*

There is a *point of inflexion at the origin*, the tangent there being $y-2z=0$.

The generator θ meets the section where

$$y=k\tan\theta,$$

$$z=k\sec\theta\tan\frac{\theta}{2}=\frac{k\tan\theta}{1+\cos\theta}.$$

Section by $y=k$.

$2x^2z-k(x^2+z^2)+2k^2z-k^3=0.$

There is a *node* at the point $x=0$, $z=k$.

There is one finite *asymptote* $z=\frac{k}{2}$.

The curve is *symmetrical about the z-axis.*

If the double point were origin, the equation would be

$$2x^2z+k(x^2-z^2)=0.$$

The generator θ meets the section where

$$x=k\cot\theta,$$

$$z=k\operatorname{cosec}\theta\tan\frac{\theta}{2}=\frac{k}{1+\cos\theta}.$$

If the double point is origin, then

$$x=k\cot\theta,$$

$$z=-\frac{k}{1+\sec\theta}.$$

Section by $z=k$.

$(y-2k)(x^2+y^2)+k^2y=0.$

There is a *node* at the point $x=0$, $y=k$.

There is only one real *asymptote*, $y=2k$.

The curve is *symmetrical about the y-axis.*

The maximum and minimum values of y are $2k$ and 0, and the curve lies entirely between the two lines $y=0$, $y=2k$.

If the origin were transferred to the double point, the equation would be

$$y(x^2+y^2)-k(x^2-y^2)=0.$$

The generator θ meets the section where

$$x=k\cos\theta\cot\frac{\theta}{2}=k\cot\theta\,(1+\cos\theta)$$

$$y=k\sin\theta\cot\frac{\theta}{2}=k(1+\cos\theta)$$

If the double point is origin, then

$$x=k\cos\theta\cot\frac{\theta}{2},$$

$$y=k\cos\theta.$$

Section by $lx+my+nz-p=0$.

If the origin is the point of intersection of the plane and the direct axis, and if the axis of x is taken perpendicular to the direct axis, the equation of the section is

$$n^2\left[lx^3-(mn+2k^2)x^2y+lxy^2-(mn+2n^2k^2)y^3\right]$$
$$-2npk.\left[x^2-lxy+n(m+n)y^2\right]+p^2(lx-mny)=0,$$

where $\qquad k^2=1-n^2.$

The co-ordinates of the double point are then

$$x=\frac{p}{k}\cdot\frac{l}{m+n},\qquad y=-\frac{p}{k}\cdot\frac{m}{n(m+n)}.$$

If the origin is transferred to the double point, the equation becomes

$$k(m+n).\left[lx^3-(mn+2k^2)x^2y+lxy^2-(mn+2n^2k^2)y^3\right]$$
$$+p.\left[(l+m)x-(k^2+mn-nl)y\right]\left[(l-m)x-(k^2+mn+nl)y\right]=0.$$

If the origin is the foot of the perpendicular from the centre of the guiding circle on to the plane, and the x-axis is the straight line in the plane perpendicular to the direct axis, the equation of the section is

$$lx^3-(mn+2k^2)\,x^2y+lxy^2-(mn+2n^2k^2)y^3$$
$$+k(m-2n).px^2+(km+4nk^2-2n^3)\,p.y^2$$
$$+lp^2x+(4n^2k^2-2k^4-mn).p^2y+k(m-2nk^2).p^3=0.$$

The co-ordinates of the double point are then

$$x=\frac{p}{k}\cdot\frac{l}{m+n},\qquad y=\frac{p}{k}\cdot\frac{k^2-mn}{(m+n)}$$

The generator θ cuts the plane where

$$k.\frac{x}{p}=\frac{(l\sin\theta-m\cos\theta)\cos\frac{\theta}{2}}{(l\cos\theta+m\sin\theta)\cos\frac{\theta}{2}+n\sin\frac{\theta}{2}}.$$

$$k \cdot \frac{y}{p} = \frac{-(l \cos\theta + m \sin\theta) \cdot n \cos\frac{\theta}{2} + k^2 \sin\frac{\theta}{2}}{(l \cos\theta + m \sin\theta) \cos\frac{\theta}{2} + n \sin\frac{\theta}{2}}.$$

The sections of (2) by planes parallel to the co-ordinate planes, and by any plane are as follows :—

Section by $x=k$.

$2y^2(y+z)+k^2.(3y-2)=0.$

The *asymptotes* are $y+z=0$, $y=\pm\frac{k}{\sqrt{2}}$.

The curve is *symmetrical about the origin.*

There is a *point of inflexion at the origin*, the tangent there being $3y-z=0$.

The equation of the generator θ referred to the axes used in (2) is

$$\frac{x}{\cos\theta \cos\frac{\theta}{2}} = \frac{\sqrt{2}y}{\cos\theta \sin\frac{\theta}{2}} = \frac{\sqrt{2}z}{(2+\cos\theta)\sin\frac{\theta}{2}}.$$

Hence the generator θ cuts the section where

$$\sqrt{2}y = k \tan\frac{\theta}{2}$$

$$\sqrt{2}z = k\left(2 \tan\theta - \tan\frac{\theta}{2}\right).$$

Section by $y=k$.

$x^2z-3kx^2-2k^2z-2k^3=0.$

The *node* is at infinity on the z-axis.

The *asymptotes* are $z=3k$, $x=\pm\sqrt{2}k$.

The curve is *symmetrical about the z-axis.*

The curve has real points for all values of x, and for all values of z except those between $-k$ and $+3k$.

If the origin were transferred to the point $(0, -k)$, the equation would assume the simple form

$$x^2z-4kx^2-2k^2z=0.$$

The generator θ cuts the section where

$$x=\sqrt{2}\, k \cot\frac{\theta}{2}$$

$$z=k(2\sec\theta+1).$$

If the vertex $(0, -k)$ is origin, then

$$x = \sqrt{2}\, k \cot\frac{\theta}{2},$$

$$z = 2k(1 + \sec\theta).$$

Section by $z = k$.

$y(3x^2 + 2y^2) - k(x^2 - 2y^2) = 0.$

There is a *node at the origin.*

There is only one real asymptote, $y = \frac{k}{3}$.

The curve is *symmetrical about the y-axis.*

It lies entirely between the two lines $y = -k$, $y = +\frac{k}{3}$.

The generator θ cuts the curve where

$$x = \frac{\sqrt{2}k}{2\tan\theta - \tan\frac{\theta}{2}},$$

$$y = \frac{k}{2\sec\theta + 1}.$$

The forms of these various sections of the Möbius Cone are shown in Plates XIV—XVIII.

The Equation of Digits:

Being an Elementary Application of a Principle of Numerical Grouping to the Solution of Numerical Equations.

By PROF. BRAJENDRANATH SEAL.

It is easy to see that the fundamental operations of arithmetic are of the nature of equations. When we are required to add two numbers, *e.g.*, 31 and 85, the equation may be represented thus: $31+85=x$, or better $(0\,.\,10^2+3\,.\,10+1)+(0\,.\,10^2+8\,.\,10+5)=d_2\,.\,10^2+d_1\,.\,10+d_0$ where d_2, d_1, d_0 are each <10. Here, by equation of corresponding digits, $d_0=1+5=6$, $d_1=3+8=\dot{1}1$, where $\dot{1}$ stands for 1 ten $\therefore d_1=1$; and $d_2=0+0+\dot{1}$; $\therefore 31+85=116$.

In the above example, we proceed to solve the equation by the method of equating corresponding digits, from the end-digit backward. If desired, we can reverse the order, and proceed from the beginning. For example, $d_2=0+0+1$, $d_1=3+8=\dot{1}1$, $d_0=1+5=6$ $\therefore 31+85=116$.

The operation of subtraction, of course, adds nothing new.

Take a case of multiplication : 31×85.

Here $(3\,.\,10+1)(8\,.\,10+5)=\ldots\, d_4\,.\,10^4+d_3\,.\,10^3+d_2\,.\,10^2+d_1\,.\,10+d_0$ $\therefore$ by equation of corresponding digits, $d_0=1\,.\,5=5$; $d_1=3\,.\,5+8\,.\,1=\dot{2}3$ $\therefore d_1=3$; $d_2=3\,.\,8+2=\dot{2}6$, $\therefore d_2=6$; and $d_3=2$; $\therefore 31\times85=2635$. Here we proceed to equate the corresponding digits from the end backward. But we could easily reverse the order.

The ordinary method of numerical division also proceeds tentatively by equating digits, but from the beginning forward. Owing to this latter circumstance the process is supposed to be based on quantitative equivalence, though nothing more than the formal identity of digits need be implied. If we reverse the order and proceed from the end-digit backward, as will be done in this paper, all implications of quantitative equivalence are put out of court, and we consider the numbers *qua* numbers, and not *qua* quantities.

The present paper then proceeds to solve numerical equations in this wider sense by the method of equating corresponding digits, and it will be shown that ultimately the equation of digits is based on the fact that integral numbers may be so arranged as to fall into certain natural groups. I will illustrate the principle of this numerical grouping by applying it to the solution of the following numerical problems: (1) numerical division, (2) recurring decimals, (3) extraction of square roots, and (4) the solution of numerical equations of any degree to any required degree of approximation.

NUMERICAL DIVISION.

First, we assume the divisor to be a number prime to 10, the radix of the scale of notation employed.

Put the divisor in the form

$$a_p \cdot 10^p + a_{p-1} \cdot 10^{p-1} + \dots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \cdot 10^0$$

where $a_p, a_{p-1} \dots a_0$ are each < 10

As the divisor is prime to 10, a_0 is also prime to 10.

Let the dividend be

$$d_r \cdot 10^r + d_{r-1} \cdot 10^{r-1} + \dots + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10^1 + d_0 \cdot 10^0$$

where $d_r, d_{r-1}, \dots d_0$ are each < 10.

First assume the dividend to be an exact multiple of the divisor.

Put the quotient in the form

$$q_m \cdot 10^m + q_{m-1} \cdot 10^{m-1} + \dots + q_2 \cdot 10^2 + q_1 \cdot 10^1 + q_0 \cdot 10^0$$

where $q_m, q_{m-1}, \dots q_0$ are each < 10

$$\therefore (a_p \cdot 10^p + a_{p-1} \cdot 10^{p-1} + \dots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0)$$

$$(q_m \cdot 10^m + q_{m-1} \cdot 10^{m-1} + \dots + q_2 \cdot 10^2 + q_1 \cdot 10 + q_0)$$

$$= d_r \cdot 10^r + d_{r-1} \cdot 10^{r-1} + \dots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0.$$

Now equate the digits on either side, *beginning with the end-digits* (*i.e.*, the units).

$\therefore$ $a_0 q_0$ must be a number ending with the digit d_0 in the scale of 10. Here a_0 and d_0 are known.

Now a_0 being prime to 10, there is only one value of q_0 which satisfies this condition. Thus q_0 is found.

Now equate the digits in the place of 10.

On the left hand, we have $(a_0 \cdot q_1 + a_1 \cdot q_0) \cdot 10 + a_0 q_0$, omitting higher powers of 10; and on the right hand $d_1 \cdot 10 + d_0$, making the same omission.

∴ the end digit of $a_0 q_1$ + the end digit of $a_1 q_0$ + the digit in the place of 10 in $a_0 q_0$ ends with a digit equal to d_1.

From which q_1 can be at once determined, as d_1, a_1, a_0 and q_0 are known.

Similarly, equate the digits in the place of 10^2.

Comparing $(q_2 . a_0 + q_1 . a_1 + q_0 . a_2) . 10^2 + (q_1 . a_0 + q_0 . a_1) . 10 + q_0 . a_0$ on the left hand with $d_2 . 10^2 + d_1 . 10 + d_0$ on the right, we find that the end digit in $q_2 . a_0$ + the end digit in $(q_1 a_1 + q_0 a_2)$ + the digit in the place of 10 in $(q_1 a_0 + q_0 a_1)$ ends with a digit equal to d_2.

From which q_2 can be determined, as the other quantities are known.

Proceeding in this way, we get the quotient

$$q_m . 10^m + q_{m-1} . 10^{m-1} + \ldots + q_2 . 10^2 + q_1 . 10 + q_0.$$

It will be seen that q_0, q_1, q_2, q_3,...are each found from an indeterminate equation of the first degree, which gives only one value of the root less than 10.

The ordinary method of division is supposed to proceed on the principle of successively finding the nearest multiple in descending powers of 10; in other words, it is held to be a method of sharing or apportioning a given quantity in equal parts, and this is done by successive approximation. The method here given proceeds on a different basis altogether, *viz.*, the properties of numbers and their successive groupings with reference to the radix of the scale of notation, groupings which give rise to a succession of indeterminate equations in the first degree, admitting each of one solution under the condition that the root must be less than the radix of the scale of notation. Here a number is not treated as a quantity composed of equal quantities, but as a group composed of groups.

Taking the decimal scale of notation, the fundamental fact at the bottom of the process, *viz.*, the grouping of numbers prime to 10 (the radix of the scale) with reference to this radix, is illustrated below :—

Take numbers with the end-digits	1,	2,	3,	4,	5,	6,	7,	8,	9,	0
Divide by 1, the end-digits of the quotients are	1	2	3	4	5	6	7	8	9	0
…… 3, ……	7	4	1	8	5	2	9	6	3	0
…… 7, ……	3,	6,	9,	2,	5,	8,	1,	4,	7,	0
…… 9, ……	9,	8,	7,	6,	5,	4,	3,	2,	1,	0

For divisor 1, the end-digits increase by 1; for divisor 9, they decrease by 1.

For divisor 3, the end-digits decrease by 3; for divisor 7, they increase by 3.

Here $1 + 9 = 10$ (the radix); also $3 + 7 = 10$.

Numerical examples :—

Divide 4394 by 13.

Write the quotient thus :— $\dots\dots + q_3 \,.\, 10^3 + q_2 \,.\, 10^2 + q_1 \,.\, 10 + q_0$

$\therefore\ (1 \,.\, 10+3)(\dots + q_3 \,.\, 10^3 + q_2 \,.\, 10^2 + q_1 \,.\, 10 + q_0)$

$= 4 \,.\, 10^3 + 3 \,.\, 10^2 + 9 \,.\, 10 + 4$

$\therefore$ the end digit of $3q_0$ is 4

$\therefore\ q_0 = 8.$

Now compare $(q_1 \,.\, 3 + q_0 \,.\, 1) \,.\, 10 + 3q_0$ and $9 \,.\, 10 + 4.$

Taking the tens on either side, and remembering that $3 \,.\, q_0 (=24)$ contains 2 tens,

$q_1 \,.\, 3 + q_0 \,.\, 1 + 2$ has 9 for its end digit

i.e., $q_1 \,.\, 3 + 10$ ends in 9

i.e., $q_1 \,.\, 3$ ends in 9

$\therefore\ q_1 = 3.$

Now compare $(q_2 \,.\, 3 + q_1 \,.\, 1) \,.\, 10^2 + (q_1 \,.\, 3 + q_0 \,.\, 1) \,.\, 10 + 3q_0$ and 394.

Taking the hundreds on either side, and remembering that

$(q_1 \,.\, 3 + q_0 \,.\, 1) \,.\, 10 = 170,$

$q_2 \,.\, 3 + q_1 \,.\, 1 + 1$ has 3 for its end digit

i.e., $q_2 \,.\, 3 + 3 \,.\, 1 + 1$ ends in 3

$\therefore\ q_2 \,.\, 3 + 4$ ends in 3

i.e., $q_2 \,.\, 3$ ends in 9

$\therefore\ q_2 = 3.$

Next compare $(q_3 \,.\, 3 + q_2 \,.\, 1) \,.\, 10^3 + (q_2 \,.\, 3 + q_1 \,.\, 1) \,.\, 10^2 + (q_1 \,.\, 3 + q_0 \,.\, 1) \,.\, 10 + 3q_0$ and 4394, and take the thousands on either side.

$\therefore\ q_3 \,.\, 3 + 3 + 1$ ends in 4

$\therefore\ q_3 = 0.$

By inspection, $q_4 = 0$, $q_5 = 0$, and so on

$$\therefore\ \frac{4394}{13} = 338.$$

Abbreviating the process, we may put it thus :—

$$13 \left| \begin{array}{cccc} 4^{(4)} & 3^{(3)} & 9^{(2)} & 4^{(1)} \\ \hline 0^{\text{iv}} & 3^{\text{iii}} & 3^{\text{ii}} & 8^{\text{i}} \end{array} \right.$$ (puxa, where p, u, x, a, are digits in the quotient.

Proceed thus :—

$3 \,.\, a = \dot{b}4^{(1)}$, wherefore $a = 8$, and $b = 2$

i.e., $\underset{,}{3} . 8^{\text{i}} = \dot{2}4^{(1)}$, where $\dot{2}4$ stands for $2 . 10+4$

$\underset{,}{1} . 8^{\text{i}} + \dot{2} = 10$

$10+\underset{,}{3} . x = \dot{y}\ 9^{(2)}$,

wherefore $3 . x$ ends in 9, and accordingly $x=3$; $\therefore\ y=1$

i.e., $10+\underset{,}{3} . 3^{\text{ii}} = \dot{1}\ 9^{(2)}$

$\underset{,}{1} . 3^{\text{ii}} + \dot{1} = 4$

$4+\underset{,}{3} . u = \dot{v}\ 3^{(3)}$,

wherefore $3 . u$ ends in 9, and accordingly $u=3$; $\therefore\ v=1$

i.e., $4+\underset{,}{3} . 3^{\text{iii}} = \dot{1}3$

$\underset{,}{1} . 3^{\text{iii}} + \dot{1} = 4$

$4+\underset{,}{3} . p = \dot{q}\ 4^{(4)}$,

wherefore $3 . p$ ends in 0, and accordingly $p=0$; $\therefore q=0$

i.e., $4+\underset{,}{3} . 0^{\text{iv}} = \dot{0}4$

$\therefore$ the quotient is $0^{\text{iv}}\ 3^{\text{iii}}\ 3^{\text{ii}}\ 8^{\text{i}}$, *i.e.*, 338.

Divide 4394971 by 37

$$\begin{array}{r|l} \underset{,}{3}\underset{,}{7} & 4^{(7)}\ 3^{(6)}\ 9^{(5)}\ 4^{(4)}\ 9^{(3)}\ 7^{(2)}\ 1^{(1)} \\ \cline{2-2} & 0^{\text{vii}}\ 1^{\text{vi}}\ 1^{\text{v}}\ 8^{\text{iv}}\ 7^{\text{iii}}\ 8^{\text{ii}}\ 3^{\text{i}} \end{array} \text{(cmspuxa}$$

Here $\underset{,}{7} . a = \dot{b}1^{(1)}$ $\therefore\ a=3,\ b=2$

i.e., $\underset{,}{7} . 3^{\text{i}} = \dot{2}1^{(1)}$

$\underset{,}{3} . 3^{\text{i}} + \dot{2} = 11$

$11+\underset{,}{7} . x = \dot{y}\ 7^{(2)}$

wherefore $7 . x$ ends in 6. $\therefore\ x=8$, and $y=6$

i.e., $11+\underset{,}{7} . 8^{\text{ii}} = \dot{6}7$

$\underset{,}{3} . 8^{\text{ii}} + \dot{6} = 30$

$30+\underset{,}{7} . u = \dot{v}\ 9^{(3)}$,

wherefore $7 . u$ ends in 9 ; $\therefore\ u=7$, and $v=7$

i.e., $30 + \underset{,}{7} . 7^{\text{iii}} = \dot{7}9^{(3)}$

$\underset{,}{3} . 7^{\text{iii}} + \dot{7} = 28$

$28+\underset{,}{7} . p = \dot{q}\ 4^{(4)}$,

wherefore $7 . p$ ends in 6. $\therefore\ p=8$ and $q=8$

i.e., $28+\underset{,}{7} . 8^{\text{iv}} = \dot{8}4^{(\ )}$

$\underset{,}{3} . 8^{\text{iv}} + \dot{8} = 32$

$32+\underset{,}{7}\ \ s = \dot{t}\,9^{(5)}$, wherefore $7s$ ends in 7. $\therefore$ $s=1$ and $t=3$

i.e., $32+\underset{\cdot}{7}\,.\,1^{\text{v}}=\dot{3}9^{(5)}$

$\underset{\cdot}{3}\,.\,1^{\text{v}}+\dot{3}=6$

$6+\underset{\cdot}{7}\,.\,m=\dot{n}\ 3^{(6)}$,

wherefore $7\,.\,m$ ends in 7. $\therefore$ $m=1$ and $n=1$

i.e., $6+\underset{\cdot}{7}\,.\,1^{\text{vi}}=\dot{1}3^{(6)}$

$\underset{\cdot}{3}\,.\,1^{\text{vi}}+\dot{1}=4$

$4+\underset{\cdot}{7}\,.\,c=\dot{d}\ 4^{(7)}$,

wherefore $7\,.\,c=0$ $\therefore$ $c=0$, $d=0$

i.e., $4+\underset{\cdot}{7}\,.\,0^{\text{vii}}=\dot{0}\ 4^{(7)}$

$\therefore$ the quotient is $0^{\text{vii}}\ 1^{\text{vi}}\ 1^{\text{v}}\ 8^{\text{iv}}\ 7^{\text{iii}}\ 8^{\text{ii}}\ 3^{\text{i}}$, *i.e.*, 118783.

Divide 253259225 by 317

$$\begin{array}{r|l} 317 & 253259225 \\ \hline & 000798925 \end{array}$$

$$\begin{aligned}
7\,.\,5^{\text{i}} &= \dot{3}5 \\
1\,.\,5+\dot{3} &= 8 \\
8+7\,.\,2^{\text{ii}} &= \dot{2}2 \\
3\,.\,5+1\,.\,2+2 &= 19 \\
19+7\,.\,9^{\text{iii}} &= \dot{8}2 \\
3\,.\,2+1\,.\,9+8 &= 23 \\
23+7\,.\,8^{\text{iv}} &= \dot{7}9 \\
3\,.\,9+1\,.\,8+7 &= 42 \\
42+7\,.\,9^{\text{v}} &= 1\dot{0}5 \\
3\,.\,8+1\,.\,9+10 &= 43 \\
43+7\,.\,7^{\text{vi}} &= \dot{9}2 \\
3\,.\,9+1\,.\,7+9 &= 43 \\
43+7\,.\,0^{\text{vii}} &= \dot{4}3 \\
3\,.\,7+1\,.\,0+4 &= 25 \\
25+7\,.\,0^{\text{viii}} &= \dot{2}5 \\
3\,.\,0+1\,.\,0+2 &= 2
\end{aligned}$$

$\therefore$ the quotient $=$ 798925.

Next take dividends which are not exact multiples of the divisor. Here the quotient as well as the remainder have to be found.

Take an example :—

$$\begin{array}{r|l} 37 & 4391 \\ \hline 16 & 4443 \end{array}$$

Proceed as before

$$\begin{aligned} 7 . 3^{i} &= \dot{2}1 \\ 3 . 3+\dot{2} &= 11 \\ 11+7 . 4^{ii} &= \dot{3}9 \\ 3 . 4+3 &= 15 \\ 15+7 . 4^{iii} &= \dot{4}3 \\ 3 . 4+4 &= 16 \\ 16+7 . 4^{iv} &= \dot{4}4 \\ 3 . 4+4 &= 16 \end{aligned}$$

Call 16 the outstanding number.

Add 1 to the dividend, and divide by 37 as before

$$\begin{array}{r|l} 37 & 4392 \\ \hline 27 & 7416 \end{array}$$

Here 27 is the outstanding number.

Add 1 again to this dividend, and divide as before

$$\begin{array}{r|l} 37 & 4393 \\ \hline 1 & 0389 \end{array}$$

Again

$$\begin{array}{r|l} 37 & 4394 \\ \hline 12 & 3362 \end{array}.$$

The outstanding numbers are 16, 27, 1, 12, ……

The numbers are in A. P. the common difference being 11, but as soon as a term becomes > 37 (the divisor), 37 is deducted.

Similarly, 4443, 7416, 0389, 3362……are in A. P. the common difference being 2973 ; but as soon as a term becomes $> 10^4$ (there being 4 digits in the dividend), 10^4 is deducted.

Now as soon as the outstanding number becomes 0, the dividend will be an exact multiple of the divisor.

$\therefore$ if $16+11n=37m$, $4391+n$ will be an exact multiple of 37.

The equation $16+11n=37m$ has to be solved by trial, without resorting to ordinary division. Where this cannot be done, the process is of no practical use, however interesting from a theoretical point of view.

Here $m=4$, $n=12$.

$\therefore$ if 12 be added to 4391, the outstanding number will be 0, and the quotient will be

$$4443+12\,.\,2973-10^4.\,m=4443+12\,.\,2973-10^4.\,4=119$$

$$\therefore\ \frac{4391}{37}=119-\frac{12}{37}=118+\frac{25}{37}.$$

Divide 4391 by 13

$$\begin{array}{r|l} 13 & 4391 \\ \hline 1 & 1107 \end{array}$$

$3\,.\,7 = 21$

$1\,.\,7+2 = 9$

$9+3\,.\,0 = 9$

$1\,.\,0+0 = 0$

$0+3\,.\,1 = 3$

$1\,.\,1+0 = 1$

$1+3\,.\,1 = 4$

$1\,.\,1+0 = 1$

Here 1 is the outstanding number

Add 1 to the dividend

$$\begin{array}{r|l} 13 & 4392 \\ \hline 5 & 4184 \end{array}$$

Here 5 is the outstanding number

Add 1 again

$$\begin{array}{r|l} 13 & 4393 \\ \hline 9 & 7261 \end{array}$$

Here 9 is the outstanding number.

Now it will be observed that 1, 5, 9,...are in A.P.

Also 1107, 4184, 7261...are in A.P.

As we go on adding unity to the dividend, the outstanding number increases or decreases in A.P. But the outstanding number can never be greater than the divisor. When therefore in the increasing A.P. a term becomes equal to the divisor, the outstanding number is 0; in other words, the dividend is an exact multiple of the divisor. Similarly when, for any value of n, the n^{th} term of the A.P. becomes a multiple of the divisor, the outstanding number is then 0; in other words, the dividend then is an exact multiple of the divisor. In a decreasing A.P., if we reach 0, we get an exact multiple. If the n^{th} term is negative, and a multiple of the divisor, then also the outstanding number is 0, *i.e.*, for that value of n, the dividend is an exact multiple.

In the above case, 1, 5, 9,...are in increasing A.P. The next term is 13, which is equal to the divisor. Therefore, by adding 3 to the dividend 4391, we get an exact multiple of 13.

Now
$$\begin{array}{r|l} 13 & 4394 \\ \cline{2-2} 0 & 0338 \end{array}$$

$$\therefore \frac{4391}{13} = 337 + \frac{10}{13}.$$

Or thus :—

Take the A.P. 1107, 4184, 7261,......

The common difference is 3077.

The next term $= 7261 + 3077 = 10338$.

Deduct 10^4 (there being 4 digits in the dividend) from 10338, and you get the quotient 338.

$$\therefore \frac{4394}{13} = 338$$

$$\therefore \frac{4391}{13} = 337 + \frac{10}{13}.$$

Explanation of the Process and Investigation of the Common Differences.

Consider the following :—

$$\begin{array}{r|l} 37 & 4391 \\ \cline{2-2} 16 & 4443 \end{array}$$

$$\begin{array}{r|l} 37 & 4392 \\ \cline{2-2} 27 & 7416 \end{array}$$

$$\begin{array}{r|l} 37 & 4393 \\ \cline{2-2} 1 & 0389 \end{array}$$

The first operation means $37 \times 4443 = 164391$

The second " $37 \times 7416 = 274392$

The third " $37 \times 389 = 14393$

or $37 \times 10389 = 384393$

By subtraction of any of the above from the next succeeding, $37 \times 2973 = 11{,}0001$

$$\therefore \begin{array}{r|l} 37 & 0001 \\ \cline{2-2} 11 & 2973 \end{array}$$

It follows from this that if the divisor be 37, and the dividend contain 4 digits, the addition of 1 (*i.e.*, 0001) to the dividend will always increase the outstanding number by 11, and the corresponding quotient by 2973. Now consider

$$\begin{array}{r|l} 37 & 1 \\ \cline{2-2} 11 & 3 \end{array}, \quad \begin{array}{r|l} 37 & 01 \\ \cline{2-2} 27 & 73 \end{array}, \quad \begin{array}{r|l} 37 & 001 \\ \cline{2-2} 36 & 973 \end{array}, \quad \begin{array}{r|l} 37 & 0001 \\ \cline{2-2} 11 & 2973 \end{array}$$

$$\begin{array}{r|l} 37 & 00001 \\ \cline{2-2} 27 & 72973 \end{array}, \quad \begin{array}{r|l} 37 & 000001 \\ \cline{2-2} 36 & 972973 \end{array}, \quad \begin{array}{r|l} 37 & 0000001 \\ \cline{2-2} 11 & 2972973 \end{array}.$$

The common differences of the outstanding numbers, therefore, recur cyclically, as we go on increasing the number of digits in the dividend by 1 (whatever be the divisor); the common differences of the corresponding quotients, written in the reverse order, obey a law of recurrence analogous to that of recurring decimals. For example, with 37 as divisor, the common differences of the outstanding numbers are 11, 27, 36; 11, 27, 36; 11, 27, 36; and so on, as the number of digits in the dividend becomes 1, 2, 3; 4, 5. 6; 7, 8, 9; and so on. For $3n+1$ digits in the dividend, the common difference is 11; for $3n+2$ digits, 27; for $3n$ digits, 36. The quotients' difference written in the reverse order will stand thus :— $3\dot{7}9\dot{2}$, there being as many digits as there are in the dividend. It will be seen that the ratio of the common difference of the corresponding quotients to that of the outstanding numbers approaches continually to the limit $\frac{10^2}{37}$, omitting the decimal point after reducing the fraction to decimals.

The r^{th} operation will mean

$$37 \times \left\{ 4443 + (r-1) \,.\, 2973 \right\} = \left\{ 16 + (r-1) \,.\, 11 \right\} . \, 10^4 + 4391 + (r-1)$$

Now give r the lowest value for which $16 + (r-1) \,.\, 11$ is a multiple of 37.

Let $16 + (r-1) \,.\, 11 = 37t$

$$\therefore \; 37 \left\{ 4443 + (r-1) \,.\, 2973 \right\} = 37t \,.\, 10^4 + 4391 + (r-1)$$

$$\therefore \; \frac{4391 + (r-1)}{37} = 4443 + (r-1) \,.\, 2973 - t \,.\, 10^4, \text{ an integer}$$

$$\therefore \; \frac{4391}{37} = 4143 + (r-1) \,.\, 2978 - t \,.\, 10^4 - \frac{r-1}{37}.$$

From the equation $16 + (r-1) . 11 = 37t$, the lowest value of $r-1$ is 12 and then $t = 4$

$$\therefore \; \frac{4391}{37} = 4443 + 12 \,.\, 2973 - 4 \,.\, 10^4 - \frac{12}{37} = 119 - \frac{12}{37} = 118 + \frac{25}{37}.$$

∴ the quotient is 118, and the remainder 25.

To explain why the outstanding numbers as well as the corresponding quotients are in A.P., consider the operations :—

$$\begin{array}{r|l} 37 & 4391 \\ \hline 16 & 4443 \end{array} \qquad \begin{array}{r|l} 37 & 4392 \\ \hline 27 & 7416 \end{array} \qquad \begin{array}{r|l} 37 & 4393 \\ \hline 1 & 0389 \end{array}$$

In the first case, we require a multiple of 7, which has 1 for its end-digit; in the second case, a multiple of 7, which has 2 for its end-digit; in the third case, a multiple of 7, which has 3 for its end-digit. The multipliers are 3, 6, 9 which are in A.P., and the multiples required are 21, 42, 63, which are also in A.P., and the digits in the place of 10 are necessarily in A.P. Then we multiply 3, 6, 9 each by 3, and add the digits 2, 4, 6, respectively; the resulting numbers must be in A.P. These are now deducted from the same number 9, and the differences must be in A.P. These differences form the end digits of fresh multiples of 7, which have to be found. The multipliers must therefore be in A.P. As we proceed in this way, the quotients as well as the outstanding numbers must be in A.P.

Now knowing beforehand that the outstanding numbers as well as the quotients must be in A.P., we need not actually compute more than 2 terms, thus :—

$$\begin{array}{r|l} 37 & 4391 \\ \cline{2-2} 16 & 4443 \end{array} \qquad\qquad \begin{array}{r|l} 37 & 4392 \\ \cline{2-2} 27 & 7416 \end{array}$$

Hence the next outstanding number must be $27+11=38$, and the quotient $7416+2973=10389$; thus :—

$$\begin{array}{r|l} 37 & 4393 \\ \cline{2-2} 38 & 10389 \end{array}$$

Deducting 37 and 10^4 respectively from the outstanding number and the quotient, we have

$$\begin{array}{r|l} 37 & 4393 \\ \cline{2-2} 1 & 0389 \end{array}$$

Let the dividend be $4391+n$, where $n<37$, then $\begin{array}{r|l} 37 & 4391+n \\ \cline{2-2} 16+11n & 4443+n.2973 \end{array}$

Here if $16+11n$ be a multiple of 37, $4391+n$ will be an exact multiple of 37.

Let $16+11n=37m$. Where $n<37$, and $m<11$. Putting $37=3.11+4$, we have $4m=16$. $\therefore\ m=4$, $n=12$.

$$\therefore\ \begin{array}{r|l} 37 & 4391+12 \\ \cline{2-2} 37\times4 & 40119 \end{array}, \quad i.e., \quad \begin{array}{r|l} 37 & 4391+12 \\ \cline{2-2} 0 & 0119 \end{array}$$

$$\therefore\ \frac{4391}{37}=119-\frac{12}{37}=118+\frac{25}{37}.$$

To shew that 37 divided by 11 leaves 4 as remainder :—

Consider $\quad \begin{array}{r|l} 11 & 37 \\ \cline{2-2} 7 & 67 \end{array}, \qquad \begin{array}{r|l} 11 & 36 \\ \cline{2-2} 8 & 76 \end{array}$

$\therefore$ the common difference of the outstanding numbers is 1.

$\therefore$ if $7+1.r$ be divisible by 11, r will be the remainder left when 37 is divided by 11.

$$\text{Put } 7+1.r=11$$
$$\therefore\ r=4,$$
$$\text{and } 37=3.11+4.$$

The following process is more successful, but it introduces considerations of greater and less, and proceeds from the left to the right as in ordinary division :—

$$\begin{array}{r|l} 37 & \underline{4391} \\ 16 & 4443 \end{array} \ldots\ldots\ldots\ldots\ldots(\mathrm{I})$$

Now consider the following :—

$$\begin{array}{r|l} 37 & \underline{0_4 \;\; 0_3 \;\; d_2 \;\; d_1} \\ 16 & x^{iv} \; x^{iii} \; x^{ii} \; x^{i} \end{array}$$

Here $3x^{iv}$ must not be > 16; also $3x^{iv}$ + the digit in place of 10 in $7x^{iv}$ must not be > 16.

$\therefore$ the greatest value of x^{iv} is 4. We take the maximum values of x^{iv} x^{iii} x^{ii} and x^{i} and the minimum values of d_2 and d_1.

Now proceed thus :—

$$\begin{array}{ll}
3 \,.\, x^{iv} + r_1 = 16 & \therefore\; x^{iv} = 4^{iv} \text{ and } r_1 = 4 \\
7 \,.\, 4^{iv} + 12 = 40_4 & \quad x^{iv} = 4^{iv} \\
3 \,.\, x^{iii} + r_2 = 12 & \therefore\; x^{iii} = 3^{iii} \text{ and } r_2 = 3 \\
7 \,.\, 3^{iii} + \;\, 9 = 30_3 & \quad x^{iii} = 3^{iii} \\
3 \,.\, x^{ii} + r_3 = 9 & \therefore\; x^{ii} = 2^{ii} \text{ and } r_2 = 3 \\
7 \,.\, 2^{ii} + r_4 = 3d_2 & \therefore\; x^{ii} = 2^{ii},\; d_2 = 2_2 \text{ and } r_4 = 18 \\
3 \,.\, x^{i} + r_5 = 18 & \therefore\; x^{i} = 5^{i} \text{ and } r_5 = 3 \\
7 \,.\, 5^{i} + \;\, 0 = 35_1 & \quad x^{i} = 5^{i},\; d_1 = 5_1
\end{array}$$

To determine x^{iii}: $3x^{iii} + \ldots = 12$ $\therefore$ the greatest value of $x^{iii} = 3$.

To determine x^{ii}: $3x^{ii} + \ldots = 9$ $\therefore$ the greatest value of $x^{ii} = 2$.

To determine d_2: $7 \,.\, 2^{ii} + \ldots = 3d_2$. Here we have to take the lowest value of d_2.

If $d_2 = 0$, $7 \,.\, 2 + 16 = 30$, $3x^{i} + r_5 = 16$, $7x^{i} + 0 = 10r_5 + d_1$, no solution.

Next take $d_2 = 1$, $7 \,.\, 2 + 17 = 31$, $3x^{i} + r_5 = 17$, $7x^{i} + 0 = 10r_5 + d_1$, no solution.

Next take $d_2 = 1$, $7 \,.\, 2 + 18 = 32$, $3x^{i} + r_5 = 18$, $7x^{i} + 0 = 10r_5 + d_1$, here by trial $x^{i} = 5$, $r_5 = 3$, $d_1 = 5$.

$$\therefore \begin{array}{r|l} 37 & 0025 \\ 16 & 43\overline{25} \end{array} \ldots\ldots(\mathrm{II})$$

From (I) $37 \times 4443 = 164391$

From (II) $37 \times 4325 = 160025$

By subtraction $37 \times 118 = 4391 - 25$

$$\therefore \quad \frac{4391}{37} = 118 - \frac{25}{37}.$$

Or proceed thus :—

$$37 \,\underline{\big|\, 43d_2\, d_1}$$
$$\therefore\quad 01x^{\text{ii}}\, x^{\text{i}}$$

where $d_2\, d_1 > 91 - 37$. This gives $d_2 = 6$, $d_1 = 6$, $x^{\text{ii}} = 1$, and $x^{\text{i}} = 8$

$$\therefore \frac{4391}{37} = 118 + \frac{25}{37}$$

This method will succeed in every case.

Now take divisors which are not prime to 10, but are prime to 5; *i.e.*, which are divisible by 2, but not by 5.

First let the dividend be an exact multiple of the divisor.

Take the end-digits	2	4	6	8	0
Divide by 2, the end-digits are	1, 6;	2, 7;	3, 8;	4, 9;	5, 0
Divide by 4, the end-digits are	3, 8;	1, 6;	4, 9;	2, 7;	0, 5
Divide by 6, the end-digits are	2, 7;	4, 9;	1, 6;	3, 8;	5, 0
Divide by 8, the end digits are	4, 9;	3, 8;	2, 7;	1, 6;	0, 5

For any given divisor and dividend, there are two possible end-digits of the quotient, the difference between the two being in every case 5 When the end-digits of the dividend increase in A.P. (*e g.*, 2, 4, 6, 8, 0), while the divisor remains the same, the end-digits of the quotients form two series each in A.P. For example, the divisor being 2, the end-digits of the quotients are either 1, 2, 3, 4, 5, an A P. with 1 as common difference, or 6, 7, 8, 9, 0, also an A.P. with the same common difference. The divisor being 8 (which is complementary to 2 with reference to 10, the radix of the scale of notation employed), the end-digits of the quotients are either 4, 3, 2, 1, 0, an A.P. with -1 as common difference, or 9, 8, 7, 6, 5, also an A.P. with the same common difference For the divisor 4, the end-digits are either 3, 1, 4, 2, 0, or 8, 6, 9, 7, 5; both in A P. with -2 for common difference, it being observed that whenever in the first series there is a change of sign from + to $-$, 5 is added to the term to retain the positive sign, and whenever in the second series a term becomes less than 5, 5 is added. Similarly for 6, which is complementary to 4, there are two A P.'s, each with 2 for common difference, it being observed that, whenever a term exceeds 5 in the first series or 10 in the second, 5 is deducted therefrom.

Now divide 4216 by 34.

Consider the operation $$34 \,\underline{\big|\, 4^{\text{iv}}\ 2^{\text{iii}}\ 1^{\text{ii}}\ 6^{\text{i}}}$$
$$x_4\ x_3\ x_2\ x_1$$

$4 \, . \, x_1$ has 6 for its end-digit

$\therefore\ x_1 = 4$ or 9.

Let $x_1 = 4_1$

$\underset{\smile}{4} \cdot 4_1 = \dot{1}6$

$\underset{\smile}{3} \,.\, 4_1 + \dot{1} = 13$

$13 + \underset{\smile}{4}x_2$ has 1^{ii} for its end-digit

Let $x_1 = 9$

$4 \,.\, 9 = \dot{3}6$

$3 \,.\, 9 + 3 + 4x_2$ has 1 for its end-digit, which is impossible.

$\therefore\ x_1$ cannot be 9.

i.e., $\underset{\smile}{4}x_2$ has 8 for its end-digit

$\therefore\ x_2 = 2_2$ or 7_2.

Let $x_2 = 2_2$

Then $13 + \underset{\smile}{4}x_2 = \dot{2}1$

and $\underset{\smile}{3} \,.\, 2_2 + \dot{2} = 8$

Let $x_2 = 7_2$; then $13 + \underset{\smile}{4}x_2 = \dot{4}1$; $\underset{\smile}{3} \,.\, 7_2 + \dot{4} = 25$

$25 + \underset{\smile}{4}x_3$ has 2^{iii} for its end-digit, which is impossible

$\therefore\ x_2$ cannot be 7_2.

$8 + \underset{\smile}{4}x_3$ has 2^{iii} for its end-digit, *i.e.*, $\underset{\smile}{4}x_3$ has 4 for its end-digit

$\therefore\ x_3$ has 1_3 or 6_3 for its end-digit.

Let $x_3 = 1_3$; then $8 + \underset{\smile}{4}x_3 = \dot{1}2$

$\underset{\smile}{3} \,.\, 1_3 + \dot{1} = 4$

$4 + \underset{\smile}{4}x_4$ has 4^{iv} for its end-digit.

If $4 + \underset{\smile}{4}x_4$ has 4^{iv} for its end-digit, $x_4 = 0_4$ or 5_4.

If $x_4 = 0_4$, $4 + \underset{\smile}{4}x_4 = 4$, and the outstanding number is $3 \,.\, 0_4 = 0$.

If $x_4 = 5_4$, $4 + \underset{\smile}{4} \cdot x_4 = \dot{2}4$, and the outstanding number is $3 \,.\, 5_4 + \dot{2} = 17$.

Thus we have found

$$\underset{\smile}{3}\underset{\smile}{4} \,\Big|\, \frac{4^{\text{iv}}\ 2^{\text{iii}}\ 1^{\text{ii}}\ 6^{\text{i}}}{0_4\ 1_3\ 2_2\ 4_1} \qquad\qquad \begin{array}{c} \underset{\smile}{3}\underset{\smile}{4} \\ 17 \end{array} \Bigg|\, \frac{4^{\text{iv}}\ 2^{\text{iii}}\ 1^{\text{ii}}\ 6^{\text{i}}}{5_4\ 1_3\ 2_2\ 4_1}$$

i.e., $34 \times 124 = 4216$

$34 \times 5124 = 174216$

The difference between the quotients is 5000; that between the dividends, 170000, or 5000 × 34.

A similar process will apply in other cases, where the dividend is an exact multiple of the divisor.

Next let the divisor be a multiple of 5.

Suppose the dividend to be an exact multiple of the divisor.

First strike out the largest common number of zeros, if any, from the end-digits of the divisor and the dividend. If the new divisor is still divisible by 5, *i.e.*, has 5 as the end-digit, then multiply both the new divisor and the new dividend by 2, and strike out one zero from the end of each. If the new divisor still has 5 at the end, multiply both divisor and dividend by 2, and strike out one zero as before. Proceed in this way until you find that the dividend does not end in 5, *i.e.*, is not divisible by 5. Then divide by the process given in this paper.

Divide 36895 by 235.

$$235 \times 2 = 470. \qquad 36895 \times 2 = 73790.$$

$$\text{Now} \quad \begin{array}{r|l} 47 & 7379 \\ \cline{2-2} 0 & \ 157 \end{array}$$

$\therefore$ the quotient is 157.

Divide 51975 by 225.

$$225 \times 2 = 450. \qquad 51975 \times 2 = 103950.$$

$$45 \times 2 = \ 90. \qquad 10395 \times 2 = \ 20790.$$

$$\text{Now} \quad \begin{array}{r|l} 9 & 2079 \\ \cline{2-2} 0 & \ 231 \end{array}$$

$\therefore$ the quotient is 231.

GENERAL SOLUTION.

General Solution.—Let the dividend be M, and the divisor, $2^a . 5^b . N$, where N is prime to 10, and a and b, both or either, may be zero.

$$\text{Here } \frac{\text{dividend}}{\text{divisor}} = \frac{M}{2^a . 5^b . N} = \frac{M . 2^b . 5^a}{N . (10)^{a+b}} = \frac{M . 2^b . 5^a}{N} \cdot \frac{1}{10^{a+b}}$$

Find out $\dfrac{M . 2^b . 5^a}{N}$ by the new process; suppose

$$\frac{M . 2^b . 5^a}{N} = Q_1 + \frac{R_1}{N}, \text{ when } R_1 < N.$$

If the dividend is an exact multiple of the divisor, $R_1 = 0$.

$$\therefore \frac{M}{2^a . 5^b . N} = \left(Q_1 + \frac{R_1}{N}\right) \cdot \frac{1}{10^{a+b}} = \frac{Q_1}{10^{a+b}} + \frac{R_1}{N . 10^{a+b}} .$$

By inspection find out the integer in $\dfrac{Q_1}{10^{a+b}}$, and also the fractional remainder.

$$\text{Let} \quad \frac{Q_1}{10^{a+b}} = Q_2 + \frac{R_2}{10^{a+b}}, \text{ where } R_2 < 10^{a+b}$$

$$\therefore \frac{M}{2^a . 5^b . N} = Q_2 + \frac{R_2}{10^{a+b}} + \frac{R_1}{N . 10^{a+b}} . = Q_2 + \frac{R_2 . N + R_1}{N . 10^{a+b}}$$

where $R_2 N + R_1 < N . 10^{a-b}$.

$$\therefore \frac{M}{2^a . 5^b . N} = Q_2 + \frac{R_2N + R_1}{N . 10^{a+b}} = Q_2 + \frac{1}{N . 2^a\, 5^b} . \frac{(R_2N+R_1) \cdot 2^a . 5^b}{10^{a+b}}$$

$\therefore$ the quotient is Q_2, and the remainder is $\frac{(R_2N+R_1) . 2^a . 5^b}{10^{a+b}}$.

Divide 36899 by 235.

Here $\frac{36899}{235} = \frac{36899.2}{235.2} = \frac{73798}{470} . = \frac{73798}{47} . \frac{1}{10}$

$$\begin{array}{r|l} 47 & 7\,3\,7\,9\,8 \\ \hline 24 & 5\,2\,6\,3\,4 \end{array} \qquad \begin{array}{r|l} 47 & 7\,3\,7\,9\,9 \\ \hline 27 & 5\,9\,0\,1\,7 \end{array}$$

$24 + 3n = 47m$. Here $n < 47$, and m not > 3. Let $m = 3$, then $n = 39$.

$\therefore$ 73798 + 39 is divisible by 47

$\therefore$ 73790 is divisible by 47

$$\begin{array}{r|l} 47 & 7\,3\,7\,9\,0 \\ \hline & 1\,5\,7\,0 \end{array}$$

$$\therefore \frac{73798}{47} = 1570 + \frac{8}{47}$$

$$\therefore \frac{36899}{235} = \left(1570 + \frac{8}{47}\right) \cdot \frac{1}{10} = 157 - \frac{8}{10.47} = 157 + \frac{1}{47.5} \cdot \frac{8}{2} = 157 + \frac{4}{235}$$

$\therefore$ the quotient is 157, and remainder 4.

Recurring Decimals :—

Let us consider the fraction $\frac{1}{N}$.

First let N be prime to 10.

For 1 in the numerator, put 1 0000000......1.

Divide by N, and you get the recurring decimal.

Take $\frac{1}{13}$

$$\begin{array}{r|l} & 0\,0\,0\,0\,0\,0\,0\,0\,1 \,|\, 13 \\ \hline 1 & 0\,7\,6\,9\,2\,3\,0\,7\,7 \end{array}$$

$$\therefore \frac{1\,0\,0\,0\,0\,0\,0\,0\,0\,1}{13} = 7\,6\,9\,2\,3\,0\,7\,7$$

$$\therefore \frac{1 \cdot 0\,0\,0\,0\,0\,0\,0\,0\,0\,1}{13} = \cdot 0\,7\,6\,9\,2\,3\,0\,7\,7$$

$$\therefore \frac{1}{13} = \cdot\dot{0}7692\dot{3}.$$

Or better thus :—For 1 in the numerator, put ·99999999.

Divide by N, and you get the recurring decimal.

Take $\frac{1}{13}$·

$$\begin{array}{r|l} \ldots\ldots\ldots 9\,9\,9\,9\,9\,9 & 13 \\ \hline 0\,|\,0\,7\,6\,9\,2\,3 & \end{array}$$

$$i.e., \ \frac{999999}{13} = 76923$$

$$\therefore \ \frac{\cdot 999999}{13} = \cdot 076923$$

$$\therefore \ \frac{\cdot 9\,9\,9\,9\,9\,9\ \ 9\,9\,9\,9\,9\,9 \ldots\ldots}{13} = \cdot \dot{0}\,7\,6\,9\,2\,\dot{3}$$

Take $\frac{1}{37}$

$$\begin{array}{r|l} \ldots\ldots\ldots 9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9\,9 & 37 \\ \hline 0\,|\,0\,2\,7 & \end{array}$$

$$\therefore \ \frac{1}{37} = \cdot \dot{0}2\dot{7}.$$

Take $\frac{1}{131}$. For 1 put ·9 9 9 9 9...

$$\begin{array}{r|l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 9\,9\,9\,9|9\,9 & 131 \\ \hline 9\,0\,8\,3\,9\,6\,9\,4\,6\,5\,6\,4\,8\,8\,5\,4\,9\,6\,1\,8\,3\,2\,0.6\,1\,0\,6\,8\,7\,0\,2\,2\,\dot{9} & \\ 0\,9\,9\,2\,3\,6\,6\,4\,1\,2\,2\,1\,3\,7\,4\,0\,4\,5\,8\,0\,1\,5\,2\,6\,7\,1\,7\,5\,5\,7\,2\,5\,1 & \\ 8\,0\,9\,1\,6\,0\,3\,0\,5\,3\,4\,3\,5\,1\,1\,4\,5\,0\,3\,8\,1\,6\,7\,9\,3\,8\,9\,3\,1\,2\,9\,7\,7 & \\ \cdot\ \dot{0}0\,7\,6\,3\,3\,5\,8\,7\,7\,8\,6\,2\,5\,9\,5\,4\,1\,9\,8\,4\,7\,3\,2\,8\,2\,4\,4\,2\,7\,4 & \end{array}$$

Thus $\frac{1}{131} = \cdot\dot{0}076\ldots\ldots\ldots 022\dot{9}$, the number of decimal places recurring being 130.

So long as the divisor does not exceed 1000, the process here given is simpler and quicker than the ordinary process.

Now take fractions of the form $\frac{P}{N}$, where N is prime to 10.

Take the fraction $\frac{4593}{13}$

Then

$$\begin{array}{r|l} 0\,0\,0\,0\ \ 4\,5\,9\,3 & 13 \\ \hline 1\,|\,0\,7\,6\,9\,\text{“}2\,6\,6\,1\text{”} & \end{array}$$

$$\therefore \ \frac{1\cdot 0\,0\,0\,0\,4\,5\,9\,3}{1\cdot 3} = 7\,6\,9\,\text{“}2\,6\,6\,1\text{”}$$

$$\therefore \ \frac{1\,0\,0\,0\,0\,4\,5\,9\,3\,0\,0\,0\,0\,0\,0\,0}{13} = 7\,6\,9\,\text{‘}2\,6\,6\,1\text{’}\,0\,0\,0\,0\,0\,0\,0$$

Again

$$\begin{array}{r|l} & 000000001 \; | \; 13 \\ \hline 1 & 076923077. \end{array}$$

$$\therefore \quad \frac{1000000001}{13} = 76923077$$

$$\therefore \quad \frac{1000000000000000001}{13} = 76923076923077.$$

Now by subtraction

$$\begin{array}{r} 769\text{‘}2661\text{’}0000000 \\ 769\ 2307\ 6923077. \\ \hline 000\ 0353\ 3076923 \end{array}$$

$$\therefore \quad \frac{45930000000-1}{13} = 3533076923.$$

$$\therefore \quad \frac{4593}{13} = 353 \cdot 3076923 \ldots = 353 \cdot \dot{3}0769\dot{2}.$$

Abbreviate the process as follows :—

$$\frac{4593}{13} \quad \begin{array}{r|l|r|l|l} 1 & 000000000000000001 & 1 & 00004593 & 13 \\ \hline & 076923076923077 & & 0769\cdot2661\text{’} & \\ & & & 07692307 & 6923077 \\ \cline{4-5} & & & 00000353\cdot & 3076923\ldots \end{array}$$

$$\therefore \quad \frac{4593}{13} = 353 \cdot \dot{3}0769\dot{2}$$

Take $\frac{735967}{29}$

$$\begin{array}{r|l} \ldots 0000000 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0000735967 & 29 \\ \hline \ldots\ldots 41379310344827586206896551724l379\text{‘}335723\text{’} & \end{array}$$

when we get this zero the outstanding number is 1.

By subtraction

$$\begin{array}{r} \cdot 034482758620689655172413 79\text{ ‘}335723\text{’ } 000000000000000000000000000 \\ \cdot 03448275862068965517241379\ \ 310344\ \ 82758\cdot 20689655172413793103448 \\ \hline \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0000000\text{‘}025378\text{’}172413793103448275862068965 52 \end{array}$$

$$\therefore \quad \frac{735967}{29} = 25378 \cdot \dot{1}72413793103448275862068965\dot{5}$$

When the divisor is not prime to 10, proceed as follows :—

To convert $\frac{P}{N}$ into decimals, where N is not prime to 10.

Suppose $N = 2^a . 5^b . M$, where M is prime to 10

$$\frac{P}{N} = \frac{P}{2^a .\, 5^b .\, M} = \frac{P \,.\, 2^b .\, 5^a}{M\,(10)^{a+b}} = \frac{P \,.\, 2^b .\, 5^a}{M} . \frac{1}{10^{a+b}} .$$

Convert $\frac{P \,.\, 2^b\ 5^a}{M}$ into decimals as before, and then find $\frac{P \,.\, 2^b\ 5^a}{M} . \frac{1}{10^{a+b}}$.

EXTRACTION OF THE SQUARE ROOT.

First let the given number be a complete square.

Let the number contain 3 or 4 digits.

Take the number $\dot{7}2\dot{9}$. The square root contains 2 digits.

Put the square root thus:—$d_1 . 10 + d_0$

$$\therefore (d_1 . 10+d_0)^2=729$$

$\therefore$ d_0^2 has 9 for its end-digit

$$\therefore d_0=3 \text{ or } 7.$$

Let $d_0=3$.

Compare

$2d_1 d_0 . 10+d_0^2$, and 2 in 729.

Here $d_0^2=9$; it contains no tens

$\therefore$ $2d_0 . d_1+0$ has 2 for its end-digit.

$\therefore$ $6d_1+0$ has 2 for its end-digit

$\therefore$ $d_1=2$ or $d_1=7$.

Now compute the hundreds and the thousands in $d_1^2 . 10^2+2d_1 d_0 . 10+d_0^2$.

Here $d_1=2$, $d_0=3$.

The hundreds are $4+1=5$.

The digit in the place of hundred is 5, which is the outstanding number.

We require a solution which will give 7 as the outstanding number in the place of hundred.

Here we find $23^2=5, 29$.

Now compute the hundreds and the thousands in $d_1^2 . 10^2+2 d_1 d_0 . 10+d_0^2$.

Here $d_1=7$, $d_0=3$.

$\therefore$ $d_1^2 . 10^2+2d_1 d_0 . 10+d_0^2$.

$=49 . 10^2+42 . 10+49$.

The hundreds are $9+4=13$, *i.e.*, the digit in the place of hundred is 3.

The digit in the place of thousand is $4+1=5$.

$\therefore$ the outstanding number is 53

Here we find $73^2=53, 29$.

Let $d_0=7$.

Compare

$2d_1 d_0 . 10+d_0^2$, and 2 in 729.

Here $d_0^2=49$; it contains 4 tens.

$\therefore$ $2d_0 . d_1+4$, or $14 d_1 + 4$ has 2 for its end-digit.

$\therefore$ $14 d_1$ has 8 for its end-digit.

$\therefore$ $d_1=2$ or 7.

Let $d_1=2$.

$14 d_1+4=32$.

The hundreds are $d_1^2+3=4+3=7$.

There are no thousands.

The outstanding number is 7.

Here we find $27^2=7, 29$.

Let $d_1=7$.

$14 d_1+4=102$.

Now d_1^2+10
$=49+10=59$

$\therefore$ the hundreds are 9, and the thousands are 5.

The outstanding number is 59.

Here we find $77^2=59, 29$.

In other words, given that the square root contains 2 digits, and the square number ends in 29, we have 4 solutions and only 4:—

23, 27, 73, 77.

Here $23+77=100$; also $27+73=100$. Also $73-23=50$, and $77-27=50$.

These relations follow at once from the following groups:—

End-digit of square number:—	1;	4,	9,	6,	5,	0
End-digit of square root	1, 9	2, 8	3, 7	4, 6	5, 5	0, 0

Here it will be seen that the sum of the two possible end-digits of the square root for the same end-digit in the square number $=10$.

We have also previously found that when the end-digit of the divisor is an even number, then for any given end-digit in the dividend, there are two possible end-digits of the quotient, and these two end-digits differ from each other by 5.

Now when the square root consists of two digits and is $= d_1 . 10 + d_0$, we find d_0 from the end-digit of d_0^2, and d_1 from the end-digit of $2 d_0 . d_1$. Here suppose d_0 to be other than 0 or 5.

Thus d_0 has two possible values, the sum of which is $= 10$; let the values be x, $10-x$.

And for each value of d_0, there are two possible values of d_1, the difference of which is 5. Let the values be y, $y+5$ for $d_0=x$, and z, $z+5$ for $d_0=10-x$.

Then the four solutions are

$y . 10+x$, $z . 10+(10-x)$, $(y+5) . 10+x$, $(z+5) . 10+(10-x)$.

Now $\{z . 10+(10-x)\}^2-(y . 10+x)^2$ will have zero in the place of unit as well as ten, because the two terminal digits of the squares are the same;

i.e. $(z^2-y^2) . 10^2+2z . 10^2+10^2-(2zx+2yx+2x) . 10$ has 0 in the place of ten.

$\therefore$ $2zx+2yx+2x$ must end in 0;

i.e., $2x(y+z+1)$ has 0 for its end-digit.

Now by supposition x is neither 0 nor 5

$\therefore$ $2(y+z+1)$ must end in 0

$\therefore$ $y+z+1$ ends in 5 or 0.

Now y and z are each less than 5,

$\therefore$ $y+z+1=5$

$\therefore$ $y+z=4$.

$\therefore$ Hence if $y\,.\,10+x=A$, $z\,.\,10+(10-x)=50-A$, $(y+5)\cdot 10+x$ $=50+A$, and $(z+5)\,.\,10+(10-x)=100-A$.

Hence the four possible square roots are A, $50-A$, $50+A$, and $100-A$ in ascending order.

Let B_1, B_2, B_3, B_4 be the corresponding outstanding numbers. These are also in ascending order.

$$\text{Now } (50-A)^2-A^2=(B_2-B_1)\,.\,100$$
$$\therefore\ 50\,(50-2A)=(B_2-B_1)\,.\,100$$
$$\therefore\ B_2-B_1=25-A$$
$$\therefore\ B_2=B_1+(25-A).$$

$$\text{Again } (50+A)^2-A^2=(B_3-B_1)\,.\,100$$
$$\therefore\ B_3-B_1=25+A$$
$$\therefore\ B_3=B_1+(25+A).$$

$$\text{Again } (100-A)^2-A^2=(B_4-B_1)\,.\,100$$
$$\therefore\ B_4-B_1=100-2A$$
$$\therefore\ B_4=B_1+100-2A$$

$\therefore$ the square roots are A, $50-A$, $50+A$ and $100-A$, and the outstanding numbers are B_1, $B_1+(25-A)$, $B_1+(25+A)$, $B_1+(100-2A)$.

Given a square number not divisible by 5, and consisting of three or four digits, we take the last two digits, and work out the lowest possible square root for these two terminal digits by the process of numerical analysis given in this paper, as also the outstanding number for this root. We then get the other solutions (with the outstanding numbers) by the above formulæ. Or we can work out any one solution, and then if it is $A<50$, the other solutions are $50-A$, $50+A$ and $100-A$. If $A>50$, the other solutions are $A-50$, $100-A$, and $150-A$.

Take the square number 1156.

$$\begin{array}{r|l} ..\ 56 & 34,\ 16 \\ & 84,\ 66 \end{array}$$

$$4\times 4=16;$$
$$2\quad 4\,.\,d_1+1 \text{ ends in } 5$$
$$\therefore\ d_1=3 \text{ or } 8.$$

One solution 34 In this case $2\,.\,4\,.\,d_1+1=25$, and the outstanding number is $3^2+2=\ 1$

The other solutions are $50-34$, $50+34$, and $100-34$.

Arranged in ascending order, the solutions are 16, 34, 66 and 84.

Taking B_1 to be the outstanding number for 16, that for $34 = B_1 + (25 - A)$.

$\therefore$ $B_1 + 25 - A = 11$, where $A = 16$. $\therefore$ $B_1 = 2$, $B_2 = 11$, $B_3 = 2 + 25 + 16 = 43$ and $B_4 = 2 + 100 - 2 \cdot 16 = 70$. Thus $16^2 = 256$, $34^2 = 1156$; $66^2 = 4356$; and $84^2 = 7056$.

If there are 5 or 6 digits in the square number, there are three digits in the square root. If we proceed by numerical analysis, any given end-digit in the square number other than 0 or 5 will give two solutions for the end-digit of the square root. Each of these two solutions again will give two solutions for any given digit in the place of ten in the square number, and so on. Ordinarily, therefore, for any given set of three terminal numbers, in a square number consisting of 5 or 6 digits, there will be $2 \times 2 \times 2$, or 8 solutions possible. It will be found that this is the case, when the square number has an odd number for its end-digit, and the square root is therefore odd; in the case of even numbers only four solutions are possible.

In the case of odd numbers not terminating in 5, the eight possible square roots for any given set of three terminal digits in a square number are A, $250-A$, $250+A$, $500-A$, $500+A$, $750-A$, $750+A$ and $1000-A$, where A is the lowest root.

The corresponding outstanding numbers will be B, $B + \frac{1}{2}(125 - A)$, $B + \frac{1}{2}(125 + A)$, $B + (250 - A)$, $B + (250 + A)$, $B + \frac{3}{2}(375 - A)$, $B + \frac{3}{2}(375 + A)$ and $B + 2(500 - A)$. In the case of even numbers, the four solutions are A, $500-A$, $500+A$, $1000-A$, A being the lowest; and the corresponding outstanding numbers will be B, $B + (250 - A)$, $B + (250 + A)$, and $B + 2(500 - A)$.

For example

$$\begin{array}{r|llll} \ldots\ 129 & 123 & 373 & 623 & 873 \\ & 127 & 377 & 627 & 877 \end{array}$$

$123^2 = 15,129$; $127^2 = 16,129$; $373^2 = 139,129$; $377^2 = 142,129$; $628^2 = 338,129$; $627^2 = 393,129$; $873^2 = 762,129$; $877^2 = 769,129$.

$$\begin{array}{r|ll} \ldots\ 376 & 124 & 624 \\ & 376 & 876 \end{array}$$

$124^2 = 15,376$; $376^2 = 141,376$; $624^2 = 389,376$; $876^2 = 767,376$.

For numbers that are not perfect squares, the roots may be found by the application of the general method given in the following section.

NUMERICAL SOLUTIONS OF EQUATIONS OF ANY DEGREE, WITH ONE UNKNOWN QUANTITY.

The general method may be best indicated by taking an example. Solve the equation

$$x^3+4307656x=5978549.$$

$$\text{Put } x=d_2 . 10^2+d_1 . 10+d_0+d_{-1} . 10^{-1}+d_{-2} . 10^{-2}+d_{-3} . 10^{-3}$$

$$+d_{-4} . 10^{-4}+\ldots$$

$$4307656=4 . 10^6+3 . 10^5+0 . 10^4+7 . 10^3+6 . 10^2+5 . 10+6$$

$$5978549=5 . 10^6+9 . 10^5+7 . 10^4+8 . 10^3+5 . 10^2+4 . 10+9.$$

The coefficient of 10^8 is $4d_2$ on the left-hand side and 0 on the right-hand side.

$$\therefore\ 4d_2=0 \qquad \therefore\ d_2=0$$

Put $d_2=0$ in the expression for x, and equate the coefficient of 10^7 on either sides.

$$\therefore\ 4d_1=0. \qquad \therefore\ d_1=0.$$

$$\therefore\ x=d_0+d_{-1} . 10^{-1}+d_{-2} . 10^{-2}+d_{-3} . 10^{-3}+d_{-4} . 10^{-4}$$

$$+d_{-5} . 10^{-5} \text{ nearly.}$$

$$\therefore\ \left(d_0+d_{-1} . 10^{-1}+d_{-2} . 10^{-2}+\ldots+d_{-5} . 10^{-5}\right)^3$$

$$+(4 . 10^6+3 . 10^5+0 . 10^4+7 . 10^3+6 . 10^2+5 . 10+6)$$

$$\left(d_0+d_{-1}10^{-1}+d_{-2}10^{-2}+d_{-3}10^{-3}+d_{-4}10^{-4}+d_{-5}10^{-5}\right)$$

$$=5978548 . 9999 \text{ . nearly.}$$

$$\therefore\ 4d_0 . 10^6+\left(3d_0+4d_{-1}\right) . 10^5+\left(3d_{-1}+4d_{-2}\right) . 10^4+\left(7d_0+3d_{-2}+4d_{-3}\right).$$
$$10^3+\left(6d_0+7d_{-1}+3d_{-3}+4d_{-4}\right) . 10^2+\left(5d_0+6d_{-1}+7d_{-2}+3d_{-4}+4d_{-5}\right) . 10+$$
$$\left(d_0^3+6d_0+5d_{-1}+6d_{-2}+7d_{-3}+3d_{-5}+4d_{-6}\right) . 10^0+\left(3d_0^2 . d_{-1}+6d_{-1}+5d_{-2}+\right.$$
$$\left.6d_{-3}+7d_{-4}+3d_{-6}+4d_{-7}\right) . 10^{-1}+\left(3d_0d^2_{-1}+3d_0^2\ d_{-2}+6d_{-2}+5d_{-3}+6d_{-4}+\right.$$
$$\left.7d_{-5}+3d_{-7}+4d_{-8}\right) . 10^{-2}+\ldots=5978548 . 9999\ldots$$

For the coefficient of the highest power of 10 on the left-hand side, *i.e.*, for the coefficient of 10^6, put c_6; for the coefficient of 10^5, put c_5, and so on.

Now equate the coefficients of 10^6 on either side.

$4d_0$ + the digit in place of 10 in c_5 + the digit in place of 10^2 in $c_4 + \dots = 5$.

$\therefore$ $d_0 = 0$ or 1; but d_0 cannot be 0, for c_5 in that case would be $4d_{-1}$ and therefore < 36, *i.e.*, the digit in the place of 10 in c_5 cannot be > 3, and c_4 has no digit in the place of 10^2.

$\therefore$ $d_0 = 1$

$\therefore$ $5 - 4d_0 = 1$

$\therefore$ $3d_0 + 4d_{-1}$ + the digit in place of 10 in c_4 + the digit in place of 10^2 in $c_3 + \dots = 19$

$\therefore$ $4d_{-1}$ + &c. $= 16$

$\therefore$ $d_{-1} = 3$; for d_1 cannot be 2, for in that case $3d_0 + 4d_{-1} = 11$, and $c_4 = 3d_{-1} + 4d_{-2}$ which is < 42, *i.e.*, the digit in the place of 10 in c_4 cannot be > 4, and therefore $3d_0 + 4d_{-1}$ + the digit in place of 10 in c_4 + &c., must be < 19.

d_1 being $= 3$, $19 - 3d_0 - 4d_{-1} = 4$

$\therefore$ $3d_{-1} + 4d_{-2}$ + the digit in the place of 10 in c_3 + the digit in the place of 10^2 in $c_2 + \dots = 47$.

$\therefore$ $4d_{-2}$ + &c. $= 47 - 9 = 38$

$\therefore$ $d_{-2} = 9$ or less, but d_{-2} cannot be 9, for then $3d_{-1} + 4d_{-2}$ + the digit in the place of 10 in $c_3 > 47$.

And d_{-2} cannot be 7, for then $3d_{-1} + 4d_{-2}$ + the digit in the place of 10 in $c_3 + \dots < 47$.

$\therefore$ $d_{-2} = 8$ $\therefore$ $47 - 3d_{-1} - 4d_{-2} = 6$.

$\therefore$ $7d_0 + 3d_{-2} + 4d_{-3}$ + the digit in the place of 10 in c_2 + the digit in the place of 100 in $c_1 + \dots = 68$.

$\therefore$ $4d_{-3}$ + &c. $= 68 - 7 - 24 = 37$.

$d_{-3} = 9$ or less.

d_{-3} cannot be 9 or 8, for then $7d_0 + 3d_{-2}$ + &c. would be > 68, and d_{-3} cannot be 6, for the $7d_0 + 3d_{-2}$ + &c. would be < 68.

$\therefore$ $d_{-3} = 7$.

$\therefore\ 6d_0+7d_{-1}+3d_{-3}+4d_{-4}+$ the digit of 10 in c_1+ &c. $=95.$

$\therefore\ 4d_{-4}+$ &c. $=47.$

d_{-4} cannot be 11 or 10.

$d_{-4}=9.$

$\therefore\ 5d_0+6d_{-1}+7d_{-2}+3d_{-4}+4d_{-5}+$ digit in place of 10 in c_0+ digit in place of 100 in $c_{-1}=114.$

$4d_{-5}+$ &c. $=14.$

d_{-5} cannot be 3 or 2 or 1

$d_{-5}=0$

$\therefore\ d_0^3+6d_0+5d_{-1}+6d_{-2}+7d_{-3}+3d_{-5}+4d_{-6}+$ digit in place of 10 in $c_{-1}+$ digit in place of 100 in $c_{-2}=148.$

$4d_{-6}+$ &c. $=29$

$d_{-6}=2$

$\therefore\ 3d_0^2.\,d_{-1}+6d_{-1}+5d_{-2}+6d_{-3}+7d_{-4}+3d_{-6}+4d_{-7}+$ digit in place of 10 in $c_{-2}+$ &c. $=219.$

$\therefore\ 4d_{-7}+$ &c. $=41$

$d_{-7}=5$

$\therefore\ 3d_0d^2_{-1}+3d_0^2.\,d_{-2}+6d_{-2}+5d_{-3}+6d_{-4}+7d_{-5}+3d_{-7}+4d_{-8}+$ &c. $=219.$

$\therefore\ 4d_{-8}+$ &c. $=16$

$d_{-8}=2$

$\therefore\ x=1\cdot38790252\ldots$

Similarly the other equations can be solved.

In the above process, the principle of equating the digits from the end-digit backward has been abandoned; and the numbers are dealt with as in the ordinary process of division, *i.e.*, we proceed to equate the digits tentatively from the beginning. Ultimately, therefore, this method of numerical solution is based on the same principle as Horner's.

In this method, d_{-1}, d_{-2}, $d_{-3}\ldots$ need not be necessarily positive. If, in any case, the value given to d_{-r} is just greater than it should be when the numerical value of the root is arranged in the decimal scale, the next digit $d_{-(r+1)}$ may be given a suitable negative value, and the process may be continued as above.

All the real positive roots may be found in this way. A negative root may be found by substituting $-y$ for x.

In actual working, the process can be easily abbreviated.

On Rates of Variation of the Osculating Conic.

By Syamadas Mukhopadhyaya.

1. If D^r stand for $\left(\frac{d}{dt}\right)^r$, where t is any independent variable, and Q_{rs} for $D^rx\,D^sy - D^ry\,D^sx$, where x and y are given functions of t, then the equation of the osculating conic at any point (x, y) of the corresponding plane curve may be written as

$$\{(Y-y)A-(X-x)B\}^2+\Gamma\{(Y-y)Dx-(X-x)Dy\}^2$$
$$=18Q_{12}^3\{(Y-y)Dx-(X-x)Dy\}$$

where $\quad A=3Q_{12}\,D^2x-Q_{13}\,Dx \qquad B=3Q_{12}\,D^2y-Q_{13}\,Dy$

$$\Gamma=3Q_{12}\,Q_{14}-5Q_{13}^2+12Q_{12}\,Q_{23}$$

and the condition, that the osculating conic passes through six consecutive points, is $\Delta=0$ where

$$\Delta=40Q_{13}^3-45Q_{12}\,Q_{13}\,Q_{14}+9Q_{12}^2\,Q_{15}-90Q_{12}\,Q_{13}\,Q_{23}+45Q_{12}^2Q_{24}.$$

These results have been elsewhere deduced from first principles. (*Vide*—"A General Theory of Osculating Conics," *Journal, Asiatic Society of Bengal*, Vol. IV, Nos. 4 and 10, 1908.) The following method, however, is more general.

If (x, y) be any given point of the plane curve and (X, Y) any other point on it, the corresponding values of the independent variable being t and $t+\tau$, then

$$X-x=Dx\,.\,\tau+\frac{1}{2\,!}\,D^2x\,.\,\tau^2+\frac{1}{3\,!}\,D^3x\,.\,\tau^3+\&c.$$

$$Y-y=Dy\,.\,\tau+\frac{1}{2\,!}\,D^2y\,.\,\tau^2+\frac{1}{3\,!}\,D^3y\,.\,\tau^3+\&c.$$

Therefore, $\quad (Y-y)Dx-(X-x)Dy$

$$=\frac{1}{2\,!}\,Q_{12}\,\tau^2+\frac{1}{3\,!}\,Q_{13}\,\tau^3+\frac{1}{4\,!}\,Q_{14}\,\tau^4+\frac{1}{5\,!}\,Q_{15}\,\tau^5+\&c.$$

and $\quad (Y-y)D^2x-(X-x)D^2y$

$$=-Q_{12}\,\tau+\frac{1}{3\,!}\,Q_{23}\,\tau^3+\frac{1}{4\,!}\,Q_{24}\,\tau^4+\frac{1}{5\,!}\,Q_{25}\,\tau^5+\&c.$$

whence it is shewn

$$\{(Y-y)A-(X-x)B\}^2+\Gamma\{(Y-y)Dx-(X-x)Dy\}^2$$
$$-18\,Q_{12}^3\{(Y-y)Dx-(X-x)Dy\}=-\frac{1}{60}Q_{12}\,\Delta\,\tau^5+\&c.$$

Hence
$$\{(Y-y)A-(X-x)B\}^2+\Gamma\{(Y-y)Dx-(X-x)Dy\}^2$$
$$-18Q_{12}^3\{(Y-y)Dx-(X-x)Dy\}=0$$

meets the given curve at five consecutive points at (x, y), determined by $\tau^5=0$. If, however, $\Delta=0$, then $\tau^6=0$, and the point (x, y) is a *sextactic* point on the given curve.

2. If ξ, η be the co-ordinates of the centre of the osculating conic at (x, y), then it is easily shewn

$$\xi=x+\frac{3Q_{12}A}{\Gamma}\qquad \eta=y+\frac{3Q_{12}B}{\Gamma}.$$

To calculate $D\xi$ and $D\eta$, we have

$$\begin{aligned}DA&=D(3Q_{12}\,D^2x-Q_{13}\,Dx)\\&=3Q_{12}\,D^3x+2Q_{13}\,D^2x-(Q_{23}+Q_{14})Dx\\&=5Q_{13}\,D^2x-(4Q_{23}+Q_{14})Dx\end{aligned}$$

Since $Dx\,.\,Q_{23}-D^2x\,.\,Q_{13}+D^3x\,.\,Q_{12}\equiv 0$

Therefore $$D\left(\frac{A}{Q_{12}^{\frac{5}{3}}}\right)=\frac{5Q_{13}D^2x-(4Q_{23}+Q_{14})Dx}{Q_{12}^{\frac{5}{3}}}-\frac{5AQ_{13}}{3\,Q_{12}^{\frac{8}{3}}}=-\frac{\Gamma\,Dx}{3\,Q_{12}^{\frac{8}{3}}}$$

Similarly $$D\left(\frac{B}{Q_{12}^{\frac{5}{3}}}\right)=-\frac{\Gamma\,Dy}{3\,Q_{12}^{\frac{8}{3}}}$$

Again $$D\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)=\frac{3Q_{12}D\Gamma-8\Gamma Q_{13}}{3Q_{12}^{\frac{11}{3}}}=\frac{\Delta}{3Q_{12}^{\frac{11}{3}}}$$

Therefore, $$D\xi=Dx+3D\,\frac{\dfrac{A}{Q_{12}^{\frac{5}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}=Dx+3\,\frac{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}D\dfrac{A}{Q_{12}^{\frac{5}{3}}}-\dfrac{A}{Q_{12}^{\frac{5}{3}}}D\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^2}=-\frac{A\Delta}{\Gamma^2}$$

Similarly $$D\eta=-\frac{B\Delta}{\Gamma^2}.$$

If we call the locus of (ξ, η) the curve of aberrancy, and σ the arcual length of the curve of aberrancy; then

$$D\sigma=\left\{(D\xi)^2+(Dy)^2\right\}^{\frac{1}{2}}=\frac{(A^2+B^2)^{\frac{1}{2}}\Delta}{\Gamma^2}$$

So that if $\Delta = 0$, then $D\sigma = 0$, a result upon which Dr. A. Mukhopadhyaya has based an elegant interpretation of the differential equation of the general conic. (*Vide—Journal, Asiatic Society of Bengal*, Vol. LVIII, part II, page 185.)

3. If a and b be the semi-axes of the osculating conic, then it can be shewn that

$$ab = \frac{27Q_{12}^{4}}{\Gamma^{\frac{3}{2}}} = 27\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{-\frac{3}{2}}.$$

It is hence evident that $\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}$ is an invariant of the point (x, y), *i.e.*, independent of the particular independent variable t, as also of the origin and direction of the axes of co-ordinates.

$$\text{Again}\quad D(ab) = -\frac{3}{2}\cdot 27\cdot\left(\frac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{-\frac{5}{2}}\cdot D\frac{\Gamma}{Q_{12}^{\frac{8}{3}}} = -\frac{27}{2}\,\frac{Q_{12}^{3}\Delta}{\Gamma^{\frac{5}{2}}}$$

$$\text{Therefore}\qquad \frac{D(ab)}{Q_{12}^{\frac{1}{3}}} = -\frac{27}{2}\cdot\frac{\dfrac{\Delta}{Q_{1}^{4}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{\frac{5}{2}}}$$

$$\text{But}\quad \frac{D(ab)}{Q_{12}^{\frac{1}{3}}} = \frac{D(ab)}{\{(Dx)^2 + (Dy)^2\}^{\frac{1}{2}}}\cdot\frac{\{(Dx)^2 + (Dy)^2\}^{\frac{1}{2}}}{Q_{12}^{\frac{1}{3}}} = \frac{d(ab)}{ds}\cdot\rho^{\frac{1}{3}}$$

where s is arcual length, and ρ the radius of curvature of the given curve at (x, y).

Hence $\dfrac{D(ab)}{Q_{12}^{\frac{1}{3}}}$ is an invariant. Therefore also $\dfrac{\Delta}{Q_{12}^{4}}$ is an invariant.

Again if r and r_1 denote two conjugate semi-diameters of the osculating conic of which r passes through the point of contact, then

$$r^2 = (\xi - x)^2 + (\eta - y)^2 = 9Q_{12}^{2}\cdot\frac{A^2 + B^2}{\Gamma^2} = 9\cdot\frac{\dfrac{A^2 + B^2}{Q_{12}^{\frac{10}{3}}}}{\left(\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}\right)^{2}}.$$

Hence we see that $\dfrac{A^2 + B^2}{Q_{12}^{\frac{10}{3}}}$ is also an invariant of the point (x, y).

It is easily shewn from the equation of the osculating conic that

$$R^2 = r^2 + r_1^2 = 9Q_{12}^{2}\left\{\frac{A^2 + B^2}{\Gamma^2} + \frac{(Dx)^2 + (Dy)^2}{\Gamma}\right\}$$

where R is the radius of the director circle of the osculating conic.

Therefore $r_1^2 = 9Q_{12}^2 \cdot \dfrac{(Dx)^2+(Dy)^2}{\Gamma}$

To calculate $D(r^2)$ and $D(r_1^2)$,

we have $$D\,\frac{A^2+B^2}{Q^{\frac{10}{3}}} = -\frac{2\Gamma(ADx+BDy)}{3Q_{12}^{\frac{13}{3}}} = -\frac{2\Gamma C}{3Q_{12}^{\frac{13}{3}}}$$

where $ADx+BDy \equiv C$. $\dfrac{C}{Q_{12}^2}$ is evidently an invariant.

Also $$D\,\frac{(Dx)^2+(Dy)^2}{Q_{12}^{\frac{2}{3}}} = \frac{2C}{3Q_{12}^{\frac{5}{3}}}$$

Since $D\,\dfrac{Dx}{Q_{12}^{\frac{1}{3}}} = \dfrac{A}{3Q_{12}^{\frac{4}{3}}}$ and $D\left(\dfrac{Dy}{Q_{12}^{\frac{1}{3}}}\right) = \dfrac{B}{3Q_{12}^{\frac{4}{3}}}$

We have, therefore,

$$D(r^2) = 9D\left[\frac{\dfrac{A^2+B^2}{Q_{12}^{\frac{10}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}}\right] = -6Q_{12}\left(\frac{C}{\Gamma}+\frac{A^2+B^2}{\Gamma^3}\Delta\right)$$

$$D(r_1^2) = 9D.\,\frac{\dfrac{(Dx)^2+(Dy)^2}{Q_{12}^{\frac{2}{3}}}}{\dfrac{\Gamma}{Q_{12}^{\frac{8}{3}}}} = 6Q_{12}\left(\frac{C}{\Gamma}-\frac{(Dx)^2+(Dy)^2}{2\Gamma^2}\Delta\right)$$

Therefore $D(R^2) = D(r^2)+D(r_1^2) = -6Q_{12}\left\{\dfrac{A^2+B^2}{\Gamma^5}+\dfrac{(Dx)^2+(Dy)^2}{2\Gamma^3}\right\}\Delta$

4. If θ be the angle which an axis of the osculating conic makes with the x-axis, then it is easily seen that

$$\tan 2\theta = \frac{2AB+2\Gamma DxDy}{A^2-B^2+\Gamma\{(Dx)^2-(Dy)^2\}} = \frac{M}{N}$$

where

$$M = \frac{2AB+2\Gamma DxDy}{Q_{12}^{\frac{10}{3}}}. \quad N = \frac{A^2-B^2+\Gamma\{(Dx)^2-(Dy)^2\}}{Q_{12}^{\frac{10}{3}}}$$

To calculate $D(\theta)$, we have

$$D\frac{AB}{Q_{12}^{\frac{10}{3}}} = -\frac{\Gamma(BDx+ADy)}{3Q_{12}^{\frac{13}{3}}} \qquad D\,\frac{DxDy}{Q_{12}^{\frac{2}{3}}} = \frac{BDx+ADy}{3Q_{12}^{\frac{5}{3}}}$$

Therefore $$DM = \frac{2\Delta DxDy}{3Q^{\frac{13}{3}}}$$

Again
$$D\frac{A^2-B^2}{Q_{12}^{\frac{10}{3}}}=-\frac{2\Gamma(ADx-BDy)}{3Q_{12}^{\frac{13}{3}}}$$

$$D\,\frac{(Dx)^2-(Dy)^2}{Q^{\frac{2}{3}}}=\frac{2(ADx-BDy)}{3Q_{12}^{\frac{5}{3}}}$$

Therefore
$$DN=\frac{\{(Dx)^2-(Dy)^2\}\Delta}{3Q_{12}^{\frac{11}{3}}}$$

so that
$$D\tan 2\theta=\frac{NDM-MDN}{N^2}\quad\text{or}\quad 2D(\theta)=\frac{NDM-MDN}{M^2+N^2}$$

But
$$NDM-MDN=\frac{2\Delta[(A^2-B^2)\ DxDy-AB\{(Dx)^2-(Dy)^2\}]}{3Q_{12}^{\frac{23}{3}}}$$

$$=\frac{2\Delta(ADx+BDy)(ADy-BDx)}{3Q_{12}^{\frac{23}{3}}}=-\frac{2\Delta C}{Q_{12}^{\frac{17}{3}}}.$$

and
$$M^2+N^2=\frac{[A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}]^2+4C^2\Gamma}{Q_{12}^{\frac{20}{3}}}$$

Therefore
$$D(\theta)=-\frac{\Delta CQ_{12}}{[(A^2+B^2)-\Gamma\{(Dx)^2+(Dy)^2\}]^2+4C^2\Gamma}$$

But
$$D\left(\frac{C}{Q_{12}^2}\right)=\frac{A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}}{3Q_{12}^3}$$

Therefore
$$D(\theta)=\frac{-\Delta CQ_{12}}{9Q_{12}^{6}\left[D\frac{C}{Q_{12}^2}\right]^2+4\,C^2\Gamma}$$

$D\theta$ becomes indeterminate, if $C=0$ and also $D\frac{C}{Q_{12}^2}=0$, which are easily shewn to be the conditions that the osculating conic is a circle. For it can be shewn that

$$D\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{-\Delta\,D\,\frac{C}{Q_{12}^2}}{6\,\Gamma^{\frac{3}{2}}}$$

which shews that $\frac{a}{b}+\frac{b}{a}$ has a minimum value when $D\frac{C}{Q_{12}^2}=0$.

But since
$$\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\frac{e^4}{1-e^2}$$

where e is eccentricity of the osculating conic, we conclude that

$$D\frac{C}{Q_{12}^2}=0$$

is the condition that the osculating conic has minimum eccentricity.

Again if δ be the angle between the line of centres and normal to the curve, then evidently

$$\tan\delta = \frac{C}{3Q_{12}^2}$$

so that $C=0$ is the condition that δ vanishes.

It may be pointed out here, in passing, that the apparent way of interpreting the singularity, when the osculating conic reduces to a circle, by saying that three consecutive circles of curvature coincide, is meaningless, unless we can shew that, in the *immediate neighbourhood* of such a singularity, a circle meets the curve in five distinct points.[*] It may be shewn, from geometrical considerations, that such is not the case. In fact, such an interpretation of the singularity would imply the coincidence of an in-cyclic point with an ex-cyclic one, which is not possible.[†]

We may, however, interpret the singularity by saying that when the conic reduces to a circle, two singularities of *different* kinds coincide. These are

$$D\left(\frac{a}{b}+\frac{b}{a}\right)=0 \text{ and } \delta=0.$$

If ϕ be the eccentric angle of the osculating conic, at the point of contact, then it is easily shewn,

$$\tan 2\phi = \frac{2ab\tan\delta}{r^2-r_1^2} = \frac{2\Gamma^{\frac{1}{2}}C}{A^2+B^2-\Gamma\{(Dx)^2+(Dy)^2\}} = \frac{2\Gamma^{\frac{1}{2}}C}{3Q_{12}^{3}I\left(\dfrac{C}{Q_{12}^2}\right)}.$$

Therefore $$D(\theta) = -\frac{\Delta Q_{12}\sin^2 2\phi}{4C\Gamma}.$$

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[*] For similar remarks on the *point of undulation* by the present writer *vide* "Geometrical Theory of a Plane Non-cyclic Arc" (*Journal, Asiatic Society of Bengal*, Vol. IV, No. 8, 1908).

[†] *Vide* "New Methods in the Geometry of a Plane Arc," Bulletin, C.M.S., Vol. I, No. I.

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On the Foundations of the Theory of Surfaces.

By Ganesh Prasad.

It is known that the one-dimensional continuum

$$\left.\begin{array}{l} x=\phi(t) \\ y=\psi(t) \end{array}\right\}$$

has a centre of curvature for every point, where $\phi''(t)$ and $\psi''(t)$ are existent. This fact naturally suggests to us the following question: *In the case of the two-dimensional continuum*

$$\left.\begin{array}{l} x=\phi(u, v) \\ y=\psi(u, v) \\ z=\chi(u, v) \end{array}\right\},$$

is the existence of the second differential coefficients ϕ_{uu}, ϕ_{uv}, ϕ_{vu}, ϕ_{vv}, ψ_{uu}, *etc., sufficient for the foundation of the theory of curvature?*

The importance of this question for the future development of the Theory of Surfaces was first recognized by Professor D. Hilbert, who proposed it to me for solution. It is the object of the present note, first to discuss the suppositions (of continuity and differentiability) on which the current definitions of lines of curvature are based, and then to formulate an answer to the question referred to above.

1. It will be sufficient for the purposes of this note to consider only two definitions, *viz.*, the definitions of Monge* and of Professor L. Bianchi; † for, the other current definitions are practically copies of the one or the other.

I. *Definition of Monge.*—" A line of curvature on a surface is such a line that the normals to the surface, drawn at any two consecutive points of the line, intersect each other."

A little reflection will show that, even if we attach some meaning to the phrase *consecutive points*,—which is, strictly speaking, meaningless,—

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* " Application of Analysis to Geometry, 1807." (In French.)

† " Lessons in Differential Geometry, vol. 1, 1902." (In Italian.)

the common procedure of regarding the co-ordinates ξ, η, ζ, of the point of intersection as constants, while differentiating the sides of the equations

$$(x-\xi)+(z-\zeta)\,p=0,$$
$$(y-\eta)+(z-\zeta)\,q=0,$$

is devoid of any precise geometrical significance.

II. *Definition of Bianchi.*—"Let a line L be drawn on a surface S, and let normals to the surface be drawn at the points of L; then, in general, these normals will form a ruled surface which is not necessarily developable. In the particular case when this ruled surface is developable, L is called a line of curvature of S.".

If r denote the radius of curvature of L, it is obvious that the above definition requires that

$$x-rX,\ y-rY,\ z-rZ$$

should be the co-ordinates of a point of a continuous curve, X, Y, Z being, as usual, the direction cosines of the normal at (x, y, z). Thus *the definition of Bianchi requires the existence and continuity of the differential coefficients of the third order of* ϕ, ψ, χ.

2. In a letter[*] addressed to Professor Hilbert, who communicated it to the Royal Society of Sciences of Göttingen, I have shown that Professor Hilbert's question should be answered in the affirmative. My method consists in discovering the geometrical significance of the lines defined by

$$(ED'-FD)\left(\frac{du}{ds}\right)^2+(ED''-GD)\frac{du}{ds}\frac{dv}{ds}$$
$$+(FD''-GD')\left(\frac{dv}{ds}\right)^2=0, \quad\ldots\ldots\ldots\ldots(1)$$

where, as usual, E, F, G, D, etc., represent the fundamental quantities; the equation (1) being the differential equation of the lines of curvature according to Bianchi's definition, provided that ϕ, ψ, χ satisfy the conditions involved in that definition. This geometrical significance, to which I was led by using a theorem of Professor R. Baire, may be described as follows:—*Let P and P′ be two points, corresponding to s and* $s+s_1$, *on a line which is defined by* (1); *further let D* (s, s_1) *denote the shortest distance between the normals to the surface at P and P′. Then there exists an unenumer-*

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[*] "On the Notion of Lines of Curvature." (In German.) See *Nachrichten der k. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse,* 1904, *Heft* 3.

able (unabzählbar) and everywhere dense aggregate, such that at every point of the aggregate

$$\lim_{s_1=+0} \frac{D(s, s_1)}{s_1}=0$$

and

$$\lim_{s_1=+0} N_s(s, s_1) \text{ exists,}$$

$N_s(s, s_1)$ *being the length of the normal at the point s, intercepted between this point and the point in which* $D(s, s_1)$ *cuts the normal. The two values of* $\lim_{s_1=+0} N_s(s, s_1)$, *corresponding to the two lines defined by* (1), *are called by me the two principal radii of curvature at the point* (x, y, z), *and are given by the quadratic equation*

$$(DD''-D'^2)r^2+(ED''+GD-2FD')r+EG-F^2=0,$$

which is the same as the quadratic equation obtained from Bianchi's definition.

Space and Time.

Address delivered at Cologne on Sept. 21st, 1908, by the Late PROF. H. MINKOWSKI of Göttingen.

[Translated by Ganesh Prasad.]

GENTLEMEN,

The views on space and time which I propose to expound to you have grown in the field of Experimental Physics. It is in this that their strength lies. Their tendency is a radical one. From this moment, space by itself and time by itself shall sink in the background, and only a certain union of these two shall retain substantiality.

I should like first to explain how, starting from the generally accepted Mechanics, one may, with the help of purely mathematical considerations, arrive at altered ideas about space and time. The equations of Newtonian Mechanics show a two-fold invariance. Their form remains unchanged first, if one subjects the underlying system of space co-ordinates to an arbitrary *change of position*; secondly, if one imposes any *uniform translation* on this system. Also the zero-point of time does not play any part. People are accustomed to look upon the axioms of Geometry as settled before they feel themselves ripe for the study of the axioms of Mechanics; for this reason, these two invariances are rarely mentioned in one breath. Each of them signifies a certain group of transformations in itself for the differential equations of Mechanics. The existence of the first group is looked upon as a fundamental attribute of space. On the other hand, people delight in punishing the second group with scorn, in order to thoughtlessly pass by it to the conclusion, that it is impossible to decide from the physical phenomena whether the space, which is supposed to be at rest, is not really in a state of uniform motion. Thus the two groups follow a separate career side by side. Their thoroughly dissimilar character may have frightened people from any attempt at combining them; but it is exactly the combination of these groups which gives us a good deal to meditate upon.

Let us try to represent the situation graphically. Let x, y, z be the rectangular co-ordinates for space and let t denote time. Space and time combined always form the subject of our perception. No one has noticed a place except at a time, and a time except at a place. However, I respect the dogma that space and time have each an independent significance. I will use the word *world-point* for a space-point corresponding to a *time-point*, *i.e.*, for a system x, y, z, t. The totality of all conceivable systems x, y, z, t may be called the *world*. I could boldly chalk out four world-axes on the board. Already one of the drawn axes consists of a number of vibrating molecules and, further, makes along with the Earth a voyage in All. Thus this axis already gives us enough to reflect upon; the somewhat greater reflection connected with the axis No. 4 does not do any harm to the mathematician. In order to leave nowhere a gaping void, we imagine to ourselves that something perceptible is existent at all places and at every moment. In order to avoid using the words matter or electricity, I will use the word *substance* for this "some thing." Let us direct our attention towards the substantial point, existent in the world-point x, y, z, t, and let us imagine to ourselves that we are in a position to recognize this substantial point at any other time. The changes dx, dy, dz in the space co-ordinates of this substantial point may correspond to an element of time dt. Thus, as the picture—so to say—of the eternal life of the substantial point, we obtain a curve in the world, *i.e.*, a *world-line* whose points admit of a one-to-one correspondence with the parameter t from $-\infty$ to ∞. The whole world appears resolved into such world-lines. And I should like to say beforehand that, according to my opinion, it would be possible for the physical laws to find their fullest expression as correlations of these world-lines.

In consequence of the notions, space and time, the x, y, z totality $t=0$ and its two flanks $t > 0$ and $t < 0$ fall asunder. If, for the sake of simplicity, we keep the zero-point of space and time fixed, then the first group of Mechanics means that, corresponding to the homogeneous linear transformations of the expression

$$x^2 + y^2 + z^2$$

into itself, we may subject the x, y, z-axes in $t = 0$ to an arbitrary rotation round the zero-point. But the second group means that, without having to alter the mechanical laws, we may also replace x, y, z, t by $x - \alpha t, y - \beta t, z - \gamma t, t$, where α, β, γ are arbitrary constants. After this, the axis of time may be given a fully arbitrary direction towards the upper half-world $t > 0$. Now, what has the demand of orthogonality in space to do with this complete freedom of the axis of time upwards?

To establish the connexion, let us take a positive parameter c and consider the figure

$$c^2t^2 - x^2 - y^2 - z^2 = 1.$$

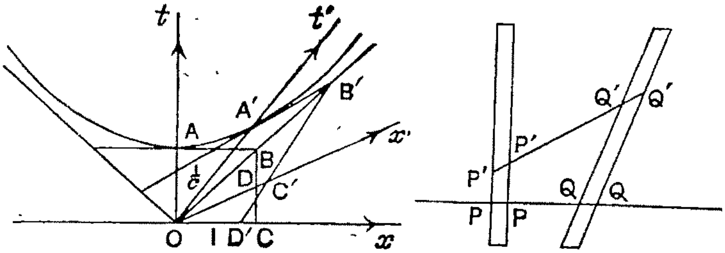

FIG. 1.

Like a hyperboloid of two sheets, it consists of two sheets separated by $t = 0$. We consider the sheet in the region $t > 0$, and we conceive now those homogeneous linear transformations of x, y, z, t into four new variables x', y', z', t', by which the expression of these sheets in the new variables becomes similar. Evidently the rotations of space about the zero-point belong to these transformations. A complete understanding of the remainder of these transformations is acquired, if we fix our eyes upon such of them as leave y and z unaltered. Let us trace (Fig. 1) the section of those sheets with the plane of the x- and t-axes, *viz.*, the upper branch of the hyperbola $c^2t^2 - x^2 = 1$, together with its asymptotes. Further, let us mark an arbitrary radius vector OA' of this hyperbolic branch from the zero-point O; lay down the tangent at A' to the hyperbola up to B', the point of intersection with the right-hand side asymptote; complete the parallelogram $OA'B'C'$; and, finally, produce $B'C'$ to D', its point of intersection with the x-axis. If we take now OC' and OA' as axes for the parallel co-ordinates x', t' with the scales $OC' = 1$, $OA' = \frac{1}{c}$, then the hyperbolic branch is again expressed by

$$c^2t'^2 - x'^2 = 1, \ t' > 0,$$

and the transition from x, y, z, t to x', y, z, t' is one of the transformations in question. We take up with these transformations the arbitrary displacements of the zero-points of space and time, and thus constitute a group of transformations which is evidently dependent on the parameter c, and which I denote by the symbol G_c.

Now let c increase indefinitely, *i.e.*, let $\frac{1}{c}$ converge to zero; then it is clear from the adjoined figure that the hyperbolic branch approaches closer and closer to the x-axis, the angle between the asymptotes becomes broader and broader, and the transformation G_c changes in the limit in such a manner that the t'-axis can have an arbitrary direction upwards and x' approaches closer and closer to x. Hence, it is clear that from G_c in the limit when c tends to ∞, *i.e.*, as the group G_∞, we have exactly the complete group belonging to the Newtonian Mechanics. In this state of affairs, and since G_c is mathematically more understandable than G_∞, a mathematician might very well, in the freedom of his imagination, come to the thought that really the phenomena of Nature possess an invariance not with the group G_∞ but with the group G_c, where c is a definite finite number, which is only extremely large in the ordinary system of units. Such a presentiment would have been an extraordinary triumph of pure Mathematics. Now, if Mathematics shows here only an unnecessary witticism, still she has the satisfaction that, thanks to her favourable antecedents, she has the power, with her senses sharpened in the exercise of free foresight, to comprehend the deep-lying consequences of such a refashioning of our conception of Nature.

I will note immediately what value of c is in question: in the place of c, the velocity of the propagation of light in empty space must make its appearance. In order not to speak of space or of emptiness, we may distinguish c as the ratio of the electrostatic and the electromagnetic units of electricity.

The subsistence of the invariance of the laws of Nature for the group G_c would be now expressed as follows:—

From the totality of natural phenomena, one may derive, by successive approximations, more and more exactly, a system of reference, x, y, z and t (*i.e.*, space and time) by means of which these phenomena can be described according to definite laws. However, this system of reference is by no means uniquely determined by the phenomena. *Corresponding to the transformations of the group G_c, we may arbitrarily vary the system of reference, without the expression of the laws of Nature being changed thereby.*

For example, in Fig. 1 we may call t' time. But then we must necessarily define space by the totality of the three parameters x', y, z; and thus the physical laws would be as exactly expressed by means of x', y, z, t' as by means of x, y, z, t. After this, we would have in the world no longer *the* space but an infinite number of spaces; just as in the three-dimensional space, there are an infinite number of planes. The three-dimensional Geometry becomes a chapter of the four-dimensional Physics. You now understand why I said at the outset that space and time shall sink in the background and only constitute a world with their union.

(II)

Now arise the questions: What circumstances compel us to adopt the altered conception of space and time? Does this conception never really disagree with phenomena? Finally, does it offer advantages for the description of phenomena?

Before we enter into these matters, I will make an important remark. If we have somehow individualized space and time, then to a substantial point at rest corresponds as world-line a straight line parallel to the t-axis; to a uniformly moving substantial point, a straight line inclined to the t-axis; and to a non-uniformly moving substantial point, a world-line curved in some manner. If we take up the world-line passing through an arbitrary world-point x, y, z, t, and find it to be there parallel to any radius vector OA' of the hyperboloidal sheet mentioned above; then we may introduce OA' as the new axis of time, and, with the new notions of space and time given thereby, the substance in the world-point in question will appear to be at rest. We will now introduce this fundamental axiom: *The substance existent in any arbitrary world-point may be always regarded as at rest, if the space and time are suitably assigned.*

This axiom means that in every world-point the expression

$$c^2\,dt^2 - dx^2 - dy^2 - dz^2$$

turns out to be always positive, or, what comes to the same thing, every velocity v turns out to be always smaller than c. Hence c is the upper limit of the velocities of all substances, and herein lies the deep significance of the quantity c. In this other form the axiom gives at first the impression of being somewhat unsatisfactory. It should, however, be considered that now we have a modified Mechanics in which the square root appears with the above differential combination of the second degree, so that cases with velocity exceeding that of light will play a part somewhat like that which figures with imaginary co-ordinates play in Geometry.

The impulse and the true motive for the acceptance of the group G_c was this, that the differential equation for the propagation of waves of light in empty space possesses that group G_c *. On the other hand the notion of rigid bodies has a sense only in a Mechanics with the group G_∞. Now, let there be an Optics with G_c and, further, let there be rigid bodies. Then it is easy to see that with the help of the hyperboloidal sheets corresponding to G_c and G_∞ a t-direction would be marked out, and this would have the following consequence: we should be able to perceive a change of phenomena with the aid of suitable rigid optical instruments in the Laboratory, when these instruments are variously set against the direction of motion of the Earth. However, all exertions having this aim in view, in particular, a famous interference experiment of Michelson, have had a negative result. In order to obtain an explanation of this fact, Lorentz built up a hypothesis the success of which lies precisely in the invariance of Optics for the group G_c. According to Lorentz every moving body should experience a contraction in the direction of motion in the ratio $1 : \sqrt{1-\frac{v^2}{c^2}}$, where v is the velocity of the body. This hypothesis sounds extremely fantastic. For, the contraction is not to be thought of as anything like the consequence of resistances in æther but purely as a present from above—as a concomitant of the circumstances of motion.

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* A true application of this fact has been already given by W. Voigt in the *Göttinger Nachrichten*, 1887, p. 41.

I will now show with our figure that the hypothesis of Lorentz is fully equivalent to the new conception of space and time by means of which it becomes much more understandable. For the sake of simplicity let us not reflect upon y and z, and let us imagine to ourselves a spacially one-dimensional world. Then a strip perpendicular like the t-axis and a strip inclined to the t-axis are respectively the images of the course of a body at rest and of that of a body in uniform motion; each of these bodies retaining a constant spacial extension. If OA' is parallel to the second strip, we may introduce t' as the co-ordinate of time and x' as the co-ordinate of space, and then the second body appears to be at rest and the first to be in uniform motion. We suppose now that the first body conceived to be at rest has the length l; in other words, the section PP of the first strip on the x-axis is equal to $l.\ OC$, where OC represents the unit of measure on the x-axis. Again we suppose that the second body conceived to be at rest has the same length l; in other words, the cross-section of the second strip parallel to the x'-axis, *i.e.*, $Q'Q'$, is equal to $l.\ OC'$. We have now these two bodies as images of two *equal* Lorentzian electrons, one at rest and the other in uniform motion. However, if we retain the original co-ordinates x, t, then the extension of the second electron is to be considered to be QQ, the section of the corresponding strip parallel to the axis of x. Now, it is evident that $QQ = l.\ OD'$ since $Q'Q' = l.\ OC'$. An easy calculation shows that, if v represent the $\frac{dx}{dt}$ for the second strip,

$$OD' = OC . \sqrt{1-\frac{v^2}{c^2}}$$

and, consequently, also

$$PP : QQ = 1 : \sqrt{1-\frac{v^2}{c^2}} .$$

And this is the sense of Lorentz's hypothesis of the contraction of electrons on account of their motion. On the other hand, if we look upon the second electron as at rest and thus adopt the system of reference x', t', then the length of the first electron will have to be denoted by the section $P'P'$ of the corresponding strip parallel to OO'. And we will find the first electron shortened in relation to the second one exactly in the ratio given above; for in the figure

$$P'P' : Q'Q' = OD : OO' = OD' : OO = QQ : PP.$$

Lorentz named the combination t' of x and t the ***place-time*** of the uniformly moving electron and appropriated a physical construction of this notion for the better understanding of the hypothesis of contraction. Nevertheless, it is the merit of A. Einstein* to have clearly recognized, that the time of one electron is as good as that of the other; in other words, that t and t' are to be treated alike. Therewith, first of all, time, as a notion uniquely determined by the phenomena, was dropped. Neither Lorentz nor Einstein shook the notion of space, perhaps for this reason that with the aforesaid special transformation, in which the x', t'-plane is congruent with the x, t-plane, an interpretation is possible as if the x-axis of space remained unchanged in its position. Also, to march over the notion of space (as in the case of the notion of time) is to be appraised as the rashness of mathematical refinement. After this further progress, which is, however, indispensable for the true understanding of the group G_c, the word *postulate of relativity*, for the demand of an invariance with the group G_c, appears to me very stale. As the sense of the postulate becomes, that only the four-dimensional world in space and time is determined by the phenomena while the projections in Space and in Time may be still taken in hand with a certain freedom, I should rather like to give this statement the name *Postulate of the absolute world* (or briefly world-postulate).

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* *Annalen der Physik*, 1905, p. 891; *Jahrbuch der Radioaktivität und Elektronik*, 1907, p. 411.

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Summary of Principal Mathematical Journals.

Mathematische Annalen, vol. 67, parts 1 and 2 (April and June, 1909).

Ph. Furtwängler, of Aachen.—The reciprocity-theorems for residues of powers with prime indices in algebraical number—corps. (In German.)

Prof. D. Hilbert was the first to extend in a comprehensive manner the theory of the quadratic reciprocity-theorems by means of the investigation of the relatively quadratic number—corps. The object of the present paper and its continuation is to complete, in a certain sense, the theory of reciprocity-theorems, in algebraical number—corps, for residues of powers with prime indices. The author attains this object by using a generalization of Hilbert's method. (p. 1—31.)

H. Weber, of Strassburg.—About the theory of cyclical number—corps. (In German.)

The author first derives some theorems from the theory of algebraica corps and of cyclical corps in an arbitrary field of rationality, and then applies these theorems to prove Kronecker's theorem that all cyclical corps in the absolute field of rationality are cyclotomic corps. (p 32—60.)

A. Wieferich, of Münster.—About the representability of numbers as sums of fifth and seventh powers of positive integers. (In German.)

The author first proves that every positive integer is representable as the sum of not more than 59 fifth powers of positive integers, and then he shows that every positive integer is representable as the sum of a fixed and finite number of seventh powers of integers. (p. 61—75.)

Leopold Fejèr, of Klausenburg.—On Laplace's series. (In German.)

In the present memoir the author first uses the arithmetical mean of the second order to obtain results relating to Laplace's series, similar to the results relating to Fourier's series (Math. Ann., vol. 58) obtained by him with the use of the arithmetic mean of the first order He then gives various applications of the new results including a proof of Weiertrass's theorem on the approximation of an arbitrary continuous function with the help of polynomials, the polynomials used by the author being Legendre's functions. The paper concludes with the formulation of some problems which, according to the author, await solution. (p. 76—109.)

Walter Schnee, of Berlin.—The identity of Cesaro's and Hölder's limiting values. (In German.)

Let a_0, a_1, a_2, etc., be an arbitrary sequence of real or complex quantities such that $\sum\limits_{n=0}^{\infty} a_n$ diverges, and put

$$s_n = S^{(0)}_n = a_0 + a_1 + \ldots + a_n,\ S'_n = s_0 + s_1 + \ldots + s_n,$$

$$\ldots,\ S_n^{(k)} = S_0^{(k-1)} + S_1^{(k-1)} + \ldots + S_n^{(k-1)} \quad ;$$

$$s_n = s_n^{(0)} = a_0 + a_1 + \ldots + a_n,\ s'_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1},$$

$$\ldots,\ s_n^{(k)} = \frac{s_0^{(k-1)} + s_1^{(k-1)} + \ldots + s_n^{(k-1)}}{n+1}$$

Then the author's result may be stated as follows:—

$$\lim_{n=\infty} \frac{k!\ S_n^{(k)}}{n^k} \quad \text{and} \quad \lim_{n=\infty} s_n^{(k)},$$

which are respectively Cesaro's limiting value and Hölder's limiting value, are always equal to one another if one of them exists. (p. 110—125.)

Peter Field, of Göttingen.—On the circuits of a plane curve. (In English.)

Defining the index of a circuit as the minimum number of real points in which it is cut by any straight line in the plane, the author proves the following theorem for curves without singular points: For every order n ($n \geqq 4$) there exist curves, $p = \frac{1}{2}(n-1)(n-2)$, formed of a single circuit of index $n-4$. (p. 126—129.)

Ernst Jacobstahl, of Berlin.—About the arithmetic of transfinite numbers. (In German.)

The present note is a supplement to the author's memoir "On the Construction of Transfinite Arithmetic," and has for its aim the settlement of the question left unsettled at the end of Art. 3 of that memoir. (p. 130—144.)

Paul Koebe, of Göttingen.—On the uniformization of the algebraical curves. (First instalment). (In German.)

By the problem of the uniformization of an analytical curve (x, y), defined by means of an arbitrary analytical function $y(x)$, is understood the problem of finding an auxiliary variable t, which is analytically dependent on x, or—what comes to the same thing—on y, and which is such that $x(t)$ and $y(t)$ are single-valued analytical functions while $y(x)$ itself is, generally speaking, an infinitely-valued function of x. This problem, which is of fundamental importance in the theory of the automorphic functions, has been carefully studied in recent years by Osgood, Brodén, Johansson and Poincairé. The object of the present memoir is to give as independent and as all-sided a treatment of the problem as possible. After an introduction, some remarks about the general notion of the linearly polymorphic uniformizing variables belonging to an algebraical curve, and a preliminary investigation relating to the uniformization of algebraical curves by means of rational functions, rational functions of an exponential function, or rational functions of an elliptic function and its derivatives, the author formulates and completely solves the problem of the determination of all linearly polymorphic uniformizing variables which have real substitution-groups; and which belong to an arbitrarily given algebraical curve. (p. 145—224.)

Hermann Weyl, of Göttingen.—On the convergence of series of orthogonal functions. (In German.)

The author first proves that, if $\phi_1(x)$, $\phi_2(x)$, etc., be a system of orthogonal functions, the series $\sum\limits_{n=1}^{\infty} c_n \phi_n(x)$ *converges essentially uniformly* when the sum $\sum\limits_{n=1}^{\infty} c_n^2 \sqrt{n}$ is existent, the following being the definition of essentially uniform convergence :—

Let $w_1(x)$, $w_2(x)$, etc., be functions of x defined for the interval $(0, 1)$. Then $\sum\limits_{n=1}^{\infty} w_n(x)$ is said to converge essentially uniformly, if, corresponding to every positive number $\epsilon < 1$, it is always possible to find an aggregate A_ϵ of measure $1-\epsilon$ and lying in the interval $(0, 1)$, such that $\sum\limits_{n=1}^{\infty} w_n(x)$ converges uniformly for all values of x belonging to A_ϵ.

He then considers various theorems allied to the above theorem which is the principal theorem of the paper. It should be noted that, throughout this paper, Lebesgue's definition of integration is used. (p. 225—245.)

L. E. J. Brouwer, of Amsterdam.—The theory of finite continuous groups, independent of the axioms of Lie. (First instalment.) (In German.)

In recent years, Professors Hilbert and Poincaré have pointed out the desirability of constructing Lie's theory of groups independently of his suppositions. The present paper is intended to contribute to such a construction. (p. 246—267.)

P. Woronetz, of Kiew.—On the rolling motion of a circular disc on an arbitrary surface under the action of given forces. (In German.)

The author first investigates a system of equations of motion for the general problem, and then applies this system to discuss certain types of motion on spheres and ellipsoids. (p. 268—280.)

Mathematische Annalen, vol. 66, part 4 (February, 1909).

This part opens with an announcement of the premature death of Professor Minkowski and of his services as a member of the editorial committee of the *Mathematische Annalen.* (In German.) (p. 417—418.)

Edmund Landau, of Berlin.—On the distribution of the zeros of Riemann's zeta-function and of those of a class of allied functions. (In German.)

The author first gives, in a simplified form, von Mangoldt's proof of the following theorem first enunciated by Riemann:—Let $N(T)$ denote the number of the zeros of the zeta-function $\zeta(s)$ whose ordinates lie between 0 and $T > 0$, 0 being excluded and T being included; further let $O(\log T)$ signify a fractional part of the order of the quantity $\log T$: then

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log(2\pi)}{2\pi} T + 0\,(\log T).$$

He then extends his procedure to the case of the function

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ stands for a character, other than the chief character, of the group of the residue-classes prime to k; k being a positive integer. The paper concludes with certain applications to the theory of prime ideals. (p. 419—445.)

Oskar Perron, of Münich.—On linear difference and differential equations. (In German.)

In the theory of the differential equations of the form

$$P_0 y + P_1 \frac{dy}{dx} + P_2 \frac{d^2y}{dx^2} + \ldots + P_n \frac{d^ny}{dx^n} = 0, \ldots\ldots (1)$$

where the coefficients P_i are polynomials in x, it is well known that, unless certain conditions are satisfied, the *general* integral does not behave in a definite manner at a multiple zero of P_n, although such may not be the case with some of the particular integrals. The object of the present paper is to apply the results of the author's recent investigations in the theory of linear difference equations to obtain more comprehensive types, than those known to previous writers, of differential equations, which are of the form (1) and for which very definite statements can be made relating to the existence of particular integrals of the kind referred to above. (p. 446—487.)

I. *Schur*, of Berlin.—On the characteristic roots of a linear substitution together with an application to the theory of integral equations. (In German.)

If $\omega_1, \omega_2, \ldots, \omega_n$ be the roots of the algebraical equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-x & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn}-x \end{vmatrix} = 0,$$

then it is known that

$$|\omega_\nu| \leqq na, \tag{1}$$

where a is the greatest of the n^2 numbers $|a_{\chi\lambda}|$ and ν has any value from 1 to n. The first of the two parts into which this paper is divided contains some new inequalities similar to (1); the most important of these being the following:—

$$\sum_{\nu=1}^{\infty} |\omega_\nu|^2 \leqq \sum_{\chi,\lambda} |a_{\chi\lambda}|^2.$$

In the second part, the author utilizes his algebraical results to show how, in the theory of the linear homogeneous integral equation

$$\lambda \int_a^b K(s, t)\phi(t)dt = \phi(s),$$

the notion of the power of a characteristic number λ may be introduced even if the kernel $K(s, t)$ be not real and symmetrical. (p. 488—510.)

Johannes Mollerup, of Copenhagen.—On the representation of an arbitrary continuous function. (In German.)

The known approximations of an arbitrary continuous function have been hitherto obtained by means of analytical materials like definite integrals, Fourier's or Taylor's series, etc. The object of the present paper is to give an axiomatic derivation of these approximations, starting from the following fundamental theorem: A sequence of continuous functions, which is uniformly convergent in an interval, has always a continuous function for its limit. (p. 511—516.)

E. Salkowski, of Charlottenburg.—About the transformation of space curves. (In German.)

This memoir is divided into four parts. It is the aim of the first part (1) to lay down all the relations between two curves which are rendered possible by means of the parallelism or orthogonality of two edges of the fundamental trihedron of these curves, (2) to determine a geometrical construction for each of these relations, and (3) to derive geometrically the characteristic equations. In the second part, the correspondence by means of parallel tangents is applied to the theory of loxodromes, with special reference to the problems of loxodromic helics and loxodromes of constant curvature. In the third part, the author investigates an application of transformation by means of orthogonal line-elements, the chief result of the investigation being the determination of a particular kind of curves of constant curvature, an infinite number of which are *algebraical*. As a simple type of such curves, the author discusses the following :—

$$x = -\frac{1}{\sqrt{1+m^2}}\left\{\frac{1-n}{4(1+2n)}\sin(1+2n)v + \frac{1+n}{4(1-2n)}\sin(1-2n)v + \tfrac{1}{2}\sin v\right\},$$

$$y = \frac{1}{\sqrt{1+m^2}}\left\{\frac{1-n}{4(1+2n)}\cos(1+2n)v + \frac{1+n}{4(1-2n)}\cos(1-2n)v + \tfrac{1}{2}\cos v\right\},$$

$$z = \frac{1}{4m\sqrt{(1+m^2)}}\cos 2n\,v,$$

n being any rational number. The fourth part contains a general investigation of those curves which are the geodesics on the tangent-surfaces of helics. (p. 517—557.)

Karl Kommerell, of Stuttgart.—Purely geometrical foundation of the theory of proportions and of area. (In German.)

Grassmann was the first to attempt to found the theory of proportions without introducing considerations of continuity, *i.e.*, without introducing irrational numbers. Among the subsequent writers, who have discussed this problem are Hoppe, H. Weber and Hilbert. But unlike Grassmann, Hoppe and Weber, Professor Hilbert does not make use of considerations of space. The aim of the present paper is first to build up the theory of proportions without using considerations of space and also without using a certain distance-calculation which Hilbert uses, and, secondly, to point out such changes in Hilbert's treatment of the theory of area as will free it from this distance-calculation. (p. 558—574.)

Georg Scheffers, of Charlottenburg.—On the isogonal surfaces of a bundle of rays. (In German.)

The author points out that the surfaces investigated by Landsberg in his paper in Part 2 of the *Annalen* (see pp. 45 and 46 of the April number of the ***Bulletin***) were first published by Scheffers in the Proceedings of the Royal Society of Sciences of Leipsic, in 1902, as special cases of a general surface. (p. 575.)

Societies and Academies.

Academy of Sciences, Paris, 1908.

July—December.

Emile Borel.—On the Analysis of Polymorphic Curves.

Georges Meslin.—On the orientation of an Anisotropic Ellipsoid in a uniform field.

M. Sanielevici.—On the differential equation of Vibrating Membranes.

A. Myller.—On a problem relating to the theory of partial differential equations of the hyperbolic type.

M. Jacob.—On a new integrometer (the apparatus described on May 11, 1908, allows the integration of the equation

$$y' = Ay^3 + By^2 + Cy + D,$$

if a particular solution is known. By slightly modifying the apparatus, the latter restriction is removed).

E. Maillet.—(*a*) On Certain Systems of Differential Equations.

„ (*b*) On differential equations and systems of reservoirs.

A. Denjoy.—Canonical products of infinite genus.

G. Darboux.—(*a*) On a problem relating to the theory of orthogonal systems and the method of the mobile trihedron.

„ (*b*) On the determination of the triple orthogonal systems comprising a family of Dupin Cyclids, and more generally, a family of surfaces with lines of Curvature plane in the two systems.

P. Cousin.—On periodic functions.

E. Nörlund.—On the Convergence of Continued Fractions.

L. Remy.—(*a*) On the value of the invariant ρ for a class of Algebraic Surfaces.

„ (*b*) On the Algebraic Surfaces which represent pairs of points of the curve of genus three.

„ (*c*) On the number of double integrals of the Second Species of certain Algebraic Surfaces.

E. Picard.—On the influence of isolated multiple-points on the number of double integrals of the Second Species of an algebraic Surface.

J. Haag.—On the Geometrical Applications of certain remarkable movements.

M. Tzitzéica.—On Conjugate net works with equal invariants.

A. Demoulin.—On the Cyclid of Lie.

L. Féjer.—On a Method of M. Darboux.

T. Lalesco.—On a class of linear differential equations of infinite order.

Jules Drach.—On Geodesic lines.

E. Traynard.—On the Condition that seven right lines should be situated on a surface of the fourth degree.

German Association of Mathematicians, 1908.

H. Minkowski of Göttingen.—Space and Time.

G. Hamel of Brünn.—On the foundations of mechanics.

E. Timerding of Strassburg.—The historical development of the notion of force.

P. Stäckel of Karlsruhe.—Special motions of a top.

R. von Mises of Brünn.—Problems of technical hydromechanics.

F. Müller of Dresden.—On plans for the edition of Euler's works.

H. Wiener of Darmstadt.—About the geometry of binary forms.

H. Reissner of Aachen.—Scientific problems in the technics of flying machines.

H. Jung of Marburg.—On algebraical functions of two independent variables.

Royal Society, London, 1909.

April—June.

A. H. Gibson.—On the depression of the Filament of maximum velocity in a stream flowing through an open channel.

J. H. Shaxby.—On the graphical determination of Fresnel's integrals.

T. H. Havelock.—On the wave-making resistance of ships—a theoretical and practical analysis.

London Mathematical Society, 1909.

April—June.

F. Tavani.—On the principles of the general theory of integral functions.

H. R. Hasse.—On the equations of electro-dynamics and the null influence of the Earth's motion on optical and electrical phenomena.

G. N. Watson.—On the Solution of a certain transcendental equation.

H. Bateman.—On the physical applications on certain conformal transformations of a space of four dimensions and the representation of a space-time point by means of a sphere.

A. E. Western.—On some criteria for the residue of eighth and other powers.

W. H. Young.—On the discontinuities of a function of one or more real variables.

H. W. Turnbull.—On Ternary quadratic Types.

J. G. Leathem.—On Gauss's theorem and on the Semi-Convergence of certain force integrals in the theory of attractions.

J. E. Littlewood.—On the continuity or discontinuity of a function defined by an infinite product.

F. J. W. Whipple.—On the behaviour at the poles of a Series of Legendre's functions representing a function with infinite discontinuities.

W. H. Salmon.—On an analogue of Pascal's theorem in three dimensions.

A. L. Dixon.—On some Symbolical expressions for the eliminant of two binary quantics.

Reviews.

Introduction to Higher Algebra.—By MAXIME BÖCHER, Professor of Mathematics in Harvard University. Pp. 321. Price 8*s*. The Macmillan Company, 1907.

Dr. Böcher is one of the fairly numerous Americans who are thoroughly in touch with the mathematical progress in Germany and France. And, if we remember right, he is one of the distinguished pupils of Prof. F. Klein and a laureate of the Royal Society of Sciences of Göttingen. For these reasons alone, his book on such a familiar subject as Algebra is sure to be of considerable interest to English-knowing mathematicians.

This book is intended for "students who have had two or three years' training in the elements of higher Mathematics, particularly analytic geometry and the calculus." Its object is "to introduce the student to higher Algebra in such a way that he shall, on the one hand, learn what is meant by a proof in Algebra and acquaint himself with the proofs of the most fundamental facts, and, on the other, become familiar with many important results of Algebra which are new to him. The book being thus intended, not as a compendium, but really, as its title states, only as an *introduction* to higher Algebra, the attempt has been made throughout to lay a sufficiently broad foundation to enable the reader to pursue his further studies intelligently, rather than to carry any single topic to logical completeness." Thus the author has found it proper to omit such important subjects as Galois' theory and the theory of numbers.

The first seven chapters deal with the most fundamental properties of polynomials, a few properties of determinants, the theory of linear dependence, linear equations, some theorems concerning the rank of a matrix, linear transformations and the combination of matrices, and the first principles of invariants. In chapters VIII—XIII, the subject of quadratic forms is very ably discussed. The next six chapters treat of polynomials in general, factors and common factors of polynomials in one variable, factors of polynomials in two or more variables, general theorems on integral rational invariants, symmetric polynomials, and polynomials symmetric in pairs of variables. The last three chapters form a good introduction to the subject of elementary divisors.

It will be seen from the above synopsis that Prof. Böcher's book is the only book on Algebra in English from which one can study the elements of some of the modern algebraical theories. This book is, therefore, bound to come into use in every University where English is the chief medium for the study of higher Mathematics.

G. PRASAD.

"An Introduction to the Study of Integral Equations."—By MAXIME BÖCHER, B.A., PH.D., Professor of Mathematics in Harvard University. Pp. 71. Price 2*s*. 6*d*. University Press, Cambridge (England), 1909.

During the last six years, integral equations, which came into prominence with the publication of a paper of the Swedish mathematician, Dr. Ivar Fredholm, in 1900, have been so frequently written upon by great mathematicians like Professors Hilbert, Picard and Poincaré, and by their pupils, that there is at present an important and fairly comprehensive theory of these equations. It is, however, unfortunate for English-knowing students of Mathematics that this theory has hardly received any contribution from English sources. It was, therefore, a happy thought of Drs. Whittaker and Leathem, the editors of the Cambridge Tracts, to enlist the services of Prof. Maxime Böcher for writing the tract under review.

The author restricts himself to the two chief types of integral equations, viz.,

$$f(x) = \int_a^x K(x, \xi)\, u(\xi)\, d\xi$$

and

$$u(x) = f(x) + \int_a^x K(x, \xi)\, u(\xi) d\xi,$$

in which $f(x)$ and $K(x, \xi)$ are to be regarded as known functions and $u(x)$ is the function to be determined. The tract contains fourteen articles besides a brief introduction and an index. The first five articles give some preliminary propositions and definitions, Abel's mechanical problem, solution of Abel's integral equation, Liouville's introduction of the integral equation of the second kind, and the method of successive substitutions. The sixth article deals with Prof. Volterra's treatment of equations of the second kind. Articles 7 and 8 are devoted to Fredholm's investigations. In Articles 9—12 the author gives a brief outline of the most interesting part of the theory, *viz.*, the part containing the results of Hilbert, Kneser and Schmidt relating to the application of the integral equation

$$u(x)=f(x)+\lambda\int_a^b K(x,\ \xi)\ u\ (\xi)d\xi$$

(in which λ is a parameter) to the problem of the expansion of an arbitrary function in an infinite series of normal functions. The last two articles are mainly devoted to Volterra's researches relating to the integral equation of the first kind.

From the above synopsis, it will be clear to those, who are already acquainted with the literature of the subject, that the author's aim, viz., "to present the main portions of the theory of integral equations in a readable and, at the same time, accurate form" and "to furnish the careful student with a firm foundation which will serve adequately as a point of departure for further work in this subject and its applications," has not been completely realized. For there are some important omissions, specially in connection with the applications of the subject. For example, the applications to the calculus of variations and to mathematical physics do not receive any mention, although it is to those applications that the theory owes its present developed form.

As we have indicated in the beginning of this review, Prof. Böcher's tract will be of considerable help to English-knowing students. Its usefulness would have been greater, had the author added some articles to the applications of the subject and given a bibliography.

G. Prasad.

"**The Mathematical Theory of Electricity and Magnetism.**—By Prof. J. H. Jeans, M.A., F.R.S. Published by the Cambridge University Press, 1908. Price 15*s.*

In the author's words "the book is written for the student and for the physicist of limited mathematical attainments." What a student generally finds difficult in these studies is not the physical but the mathematical part. The author has adopted the plan of sharpening the tools what he requires next. We are not sure how far this method succeeds in removing the mathematical difficulties.

The book is intended to be an introduction to the study of Maxwell's theory. Unlike the modern books (for instance of Abraham, or Richarz) it starts with Coulomb's law and arrives at the general equations in the fifteenth chapter. Nowadays one expects to find the use of Vector analysis in such treatises, and that would have diminished the size of the book to a certain extent.

The first five chapters deal with electrostatics and cover almost the same grounds as in the well-known Elements of Sir J. J. Thomson. All these are based on the assumption of action at a distance. In the sixth chapter we find a theory of explaining electrostatic forces with a tension $\frac{R^2}{8\pi}$ per unit area in the direction of the lines of force and a pressure $\frac{R^2}{8\pi}$ per unit area perpendicular to the lines of forces in the ether.

The next chapter is dedicated to the general analytical theorems with some of their applications to electrostatics.

In dealing with the problems of electrostatic Induction we generally take to the method of Images or to Spherical Harmonics. In the seventh chapter we find some well-known problems solved with the first method. Thirty-five pages are devoted to Spherical Harmonics, with its electrostatic applications (those who want more information may refer to the now published lectures of Helmholz edited by Krigar Menzel). The two dimensional problems form an interesting part of this chapter. The ninth chapter brings us to the steady currents in linear conductors and their laws. A general theory of a network is given. The next chapter deals with steady currents in continuous media.

The eleventh chapter brings us to permanent magnetism. The mathematical theorems are worked out In the next chapter we come to induced magnetism. The molecular theories are also dealt with. We miss here, however, a discussion about the modern views of permanent magnetism as professed by Prof. Cohn and his students. In the thirteenth chapter we have the magnetic field produced by electric currents, and in the next induction of currents in linear circuits. In page 435 we find the energy of the field calculated where the coefficients of induction are introduced as mathematical coefficients. The conditions under which these equations hold good should have been mentioned. We should have liked to see the coefficients of induction defined also from the physical standpoint, even in case where we have iron in the neighbourhood. In page 453 and the following pages we have the mathematical theory of transformers.

The general equations are arrived at in the fifteenth chapter which deals with the induction of currents in continuous media. The next chapter gives us the dynamical theories and their application to electromagnetic problems.

The remaining chapters deal with electromagnetic theory proper. The excellent experiments of Hagen and Rubens are referred to. These experiments are now looked upon as one further corroboration of Maxwell's theory. At the end of each chapter References are given to works on the subject. This list is rather incomplete as it mentions practically only a few of the standard English books.

One special feature of the book is that it contains a splendid collection of examples.

S. K. Dutt.

Notes and News.

Wolfskehl Foundation; Lectures of Prof. Poincaré.—On the invitation of the Committee of the Wolfskehl Foundation, Prof. Henri Poincaré delivered a series of lectures at Göttingen from 22nd to 28th April, 1909. The lectures principally dealt with the Integral Equations and their applications to Physics and Astronomy.

The subjects of the lectures were as follows:—

1. The theory of integral equations. 2. Application of integral equations to the theory of tides. 3. Application of integral equations to the theory of Hertzian waves. 4. Reduction of the Abelian integral. 5. On the transfinite numbers. 6. The new mechanics.

The late Prof. Minkowski of Göttingen.—At a special meeting of the Royal Society of Sciences of Göttingen held on the 1st of May last, Prof. Hilbert delivered a very eloquent and instructive address on the life and work of the late Prof. Hermann Minkowski. We learn from this address that Prof. Minkowski, a Russian Jew by birth, successfully competed for the Grand Prize of the Academy of Sciences of Paris, when he was only an 18-year-old student at the University of Königsberg. He lived and died a sage. Even on his death-bed, while suffering from the effects of an unsuccessful operation for appendicitis, his thoughts were centred on Mathematics, and he spent the last hours of his existence making arrangements for the reading of the proof-sheets of a work of his.

A Mathematico-meteorological Memoir.—Mr. Gilbert Walker, F.R.S., the head of the Meteorological Department of the Government of India, has written a notable memoir on "Correlation in Seasonal Variation of Climate" (vol. xx of the Memoirs of the Indian Meteorological Department). The memoir is confined to a deduction of the correlation coefficient, the regression equations for two or more variables, and the remaining formulæ of great importance in the theory of correlation.

Relevance of Mathematics.—In nature, May 27th, 1909, Mr. Philip E. B. Jourdain has written an interesting and instructive paper on the *logical* nature of all mathematical conceptions and methods, as opposed to Kant's view that mathematical reasoning is not strictly formal but always uses *a priori* intuitions of Space and Time. Mr. Jourdain has noticed how there appeared in the mathematics of the nineteenth century an increasing tendency towards critical examination into the validity and the limits of validity of mathematical conceptions and methods, how Gauss combined, in an unexampled degree, the power of discovery and profound critical insight, how of still greater influence was the work of Cauchy, the great French mathematician, how Cauchy may be regarded as the indirect inspirer of Weierstrass, who in his turn was the chief inspirer of Georg Cantor, and how it is to the influence of Cantor and Dedekind that we owe that line of thought which has resulted in this great discovery of the logical nature of mathematics.

The Osiris Prize.—The Institute of France has awarded the Osiris Prize, of the value of 100,000 francs, to M. Louis Blériot and M. Gabriel Voisin for their experiments and achievements in aërial navigation. The prize is awarded every three years to the most remarkable contribution to the cause of human progress during that period.

Prize Problems.—The Academy of Sciences and Letters, Denmark, has issued a circular setting subjects for prize essays, among which the one on Astronomy is as follows :—

"To examine the conditions in which it is possible to determine the mass of a comet and to investigate whether these conditions are satisfied by comets which do not traverse exactly the orbits calculated for them by the usual methods. It is required that for at least one comet of this class, the orbit should be calculated, using the whole of the seven constants of the formulæ relating to the movement of two bodies, and that the results obtained should be compared with those of observation."

The prize would be the gold medal of the Academy, valued at about £18. The paper must be sent in to the Secretary, Prof. H. G. Leuthen, the University, Copenhagen, before 31st October, 1910, and may be written in English.

The Royal Academy of Belgium announces that, in the year 1910, it will award a prize of 800 francs for systematic and didactic exposition of the recent researches on the partial differential equations of the second order, and a prize of 600 francs for an essay expounding and completing the researches made in the Calculus of Variations since 1850.

The Royal Society of Naples announces that, in the year 1910, it will award a prize of 1,000 francs for a memoir, which must contain a systematic exposition of the acquired notions on the geometrical configurations of the plane and of the spaces occurring in connection with the theory of substitutions, and which may contain some new contribution as well.

The Prince Jahlonowski Society of Leipsic announces that a prize of 1,500 marks will be awarded in 1910 for an essay advancing the theory of "fundamental superposition" in the subject of potentials, and that a prize of equal value will be awarded in 1911 for a new contribution to the theory of rainbows.

It is announced that the grand prize (of 10,000 francs) of the Academy dei Lincei of Rome has been divided between Prof. E. Enriques of Bologna and Prof. T. Levi-Civita of Padua.

New Publications.

Ch. André.—**Les planétes et leur origine.** Pp. 285. Gauthier-Villars, Paris. 10 fr.

L. Bianchi.—**Lezioni di Geometria Analitica.** Pp. 816. Pisa. 14 lires.

E. Czuber.—**Einführung in die höhere Mathematik.** Pp. 382. Leipzig, B. G. Teubner. 12 m.

Encyclopédie des Sciences Mathématiques Pures et Appliquées. Tome II, Volume 4, Fasc. 2.—Calcul des probabilités. Théorie des Erreurs. Applications diverses. Pp. 161—320. Gauthier-Villars. 6 fr. 25. Tome II, Volume 1. Fasc. 1.—Fonctions de Variables Réelles. Pp. 1—112. Gauthier-Villars, Paris.

Huyghens.—**Œuvres.** T. XI. Travaux Mathématiques. Pp. 370. Paris, Gauthier-Villars. 23 fr.

Jahrbuch über die Fortschritte der Mathematik. 37 Bd. Jahrg 1906, 2 Heft. Pp. 485—692. G. Reimer, Berlin. 7 m. 50.

G. Kowalewski. — **Grundzüge der Differential- u. Integralrechnung.** Pp. 452. Leipzig. B. G. Teubner. 12 m.

H. Poincaré.—**Leçons de Mécanique Céleste t. II, partie II^{e}. Theorie de la lune.** Pp. 137. Paris, Gauthier-Villars. 5 fr.

D. Hilbert.—**Wissenschaft u. Hypothese: VII. Grundlagen der Geometrie.** Pp. 279. Leipzig. B. G. Teubner. 6. m.

M. Schuster.—**Solar Research.** Pp. 243. London. Sherrat and Hughes. 7*s*. 6*d*.

F. Enriques.—**Lezioni di Geometria descrittiva.** (Second edition). Bologna. 12 l.

F. Enriques.—**Lezioni de Geometria proiettiva.** (Third edition). Bologna. 10 l.

A. Reymond.—**Logique et Mathématiques.** Pp. 219. Paris, F. Alcan. 5 fr.

Benchara Branford.—**A Study of Mathematical Education, including the teaching of Arithmetic.** Pp. 392. Clarendon Press, Oxford.

E. and F. Cosserat.—**Théorie des Corps deformables.** Pp. 226. Hermann, Paris. 6 fr.

Sir George Greenhill.—**Notes on Dynamics.** Pp. 221. Wyman & Sons, London. 3*s*.

G. Veronese.—**Nozioni di Geometria Intuitiva.** Pp. 123. 1 l. 50. Drucker, Padua.

G. Veronese.—**Elementi di Geometria.** Pp. 139. Drucker, Padua.

E. Borel.—**Elements de la Théorie des Probabilités.** Pp. 190. Hermann, Paris. 6 fr.

J. V. Collins.—**Practical Elementary Algebra.** Pp. 420. American Book Co., New York. 1*s*.

P. Duhem.—**Etudes sur Léonard de Vinci.** Pp. 474. Hermann, Paris. 15 fr.

F. Klein.—**ElementarMathematik vom höheren Standpunkte aus.—Teil II: Geometrie.** Pp. 515. B. G. Teubner, Leipzig.

T. Sundara Row.—**Elementary Solid Geometry. Part III.—Polyhedrons.** Pp. 91. St. Joseph's College Press, Trichinopoly, India.

C. Elliot.—**Classification and Mathematics.** Pp. 41. (Published by the Author, North St., Oundle).

H. Crabtree.—**An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion.** Pp. 140. 5*s*. 6*d*. net. Longmans & Green.

P. E. B. Jourdain.—**The Development of the Theory of Transfinite Numbers.** Reprinted from the Archiv der Mathematik and Physik.

Paul Carus.—**The Foundations of Mathematics:—A contribution to the Philosophy of Geometry.** Pp. 141. Chicago. The Open Court Publishing Company.

S. Arrhenius. **The Life of the Universe.** 2 Volumes. Harper's Library of Living Thought. 5*s*.

S. Arrhenius.—**Worlds in the Making. The Evolution of the Universe.** Harper & Brothers.

O. Lodge.—**The Ether of Space.** Harper's Library of Living Thought. 5*s*.

H. Burkhardt.—**Funktionentheoretische Vorlesungen.** 1 Bd. 1 Heft. Algebraische Analysis. Second edition. Pp. 199. Veit & Co., Leipzig. 5 m. 60.

C. Santreaux.—**Essai sur les axiomes des Mathématiques.** (Étude critique élémentaire). Pp. 78. A. Gratier and J. Rey. Grenoble.

G. F. Chambers.—**The Story of the Comets.** Pp. 256. Clarendon Press, Oxford. 6*s.*

Niels Nielson.—**Lehrbuch der Unendlichen Reihen.** Pp. 287. Teubner, Leipzig.

A. L. Crelle.—**Calculating Tables**; giving the products of every two numbers from 1 to 1,000 and their applications to the multiplication and division of all numbers above 1,000. New edition, by O. Seeliger; with tables of the square-numbers and cube-numbers from 1 to 1,000. Lemeke & Büschner, New York. 5*s.*

F. Müller.—**Führer durch die Mathematische Literatur mit besonderer berücksichtigung der historichwichtigen Schriften.** Pp. 252. Teubner, Leipzig

E. Picard.—**Traité d'Analyse.** t. III, second edition. Pp. 604. Gauthier-Villars, Paris.

E. Jouguet. **Lectures de Mécanique.** La Mécanique enseigneé par les auteurs originaux. I° partie: La Naissance de la Mécanique. II° partie: L'organisation de la Mécanique. Gauthier-Villars.

S. Reye.—**Die Geometrie der Lage.** Vortrage. 5te, verbesserte und vermehrte Auflage. Pp. 255. Kröner, Leipzig. 10 m.

A. Föppl.—**Vorlesungen über technische Mechanik.** (In 6 Volumes). Volume 4: Dynamik. Pp. 422. Teubner, Leipzig. 10 m.

E. L. Hancock.—**Applied Mechanics for Engineers.** A text-book for Engineering Students. Pp. 385. Macmillan, New York. 2*s.*

P. E. B. Jourdain.—**Lagrange, Rodrigues, Jacobi and Gauss.** Abhandlungen uber die Prinzipien der Mechanik. (Ostwald's Klassiker der exakten Wissenschaften, 167). Pp. 68. Engelmann, Leipzig. 1 m. 40.

T. L. Heath.—**The Thirteen Books of Euclid's Elements,** translated from the Text of Heiberg with Introduction and Commentary. 3 Volumes, University Press, Cambridge.

C. Prelini.—**Graphical Determination of Earth Slopes, Retaining Walls and Dams.** Constable, London. 8*s.*

J. Andrade.—**La Géométrie Naturelle.** t. I. Géométrie qualitative. E. Cornély & Co., Paris. 2 fr.